People's Democratic Republic of Algeria

Ministry of Higher Education and Scientific Research

National Higher School of Hydraulics (ENSH), Blida

Department of Preparatory Training

Course Booklet

Mechanical and Electromagnetic waves

Prepared by Dr. Badis RIAH

Academic Year 2024-2025

Preface

This course booklet is intended for second-year students in scientific and technical preparatory classes, as well as students in the ST-SM university programs. It offers a comprehensive course in Physics 4 on mechanical and electromagnetic waves, accompanied by exercises and solved problems to help students strengthen their skills. Aligned with the official curriculum, it serves as a preparation guide for exams and entrance competitions to top engineering schools.

The booklet is structured into several chapters, first covering mechanical waves, starting with general concepts of waves, followed by the study of the vibrating string and acoustic waves in fluids, before addressing elastic waves in solids. Each chapter concludes with a series of exercises to reinforce key concepts.

Next, it explores electromagnetic waves, beginning with their propagation in a vacuum, including a review of Maxwell's equations and the study of wave structure, before examining their behavior in different media, particularly plasmas, conductors, and perfect conductors. Finally, the propagation of electromagnetic waves in waveguides is studied in detail.

The course booklet concludes with a series of exercises specifically focused on electromagnetic waves, allowing students to deepen their understanding and effectively prepare for assessments.

Content

Chapter I: Generalities on waves

I-1- Definition of wave

A mechanical wave is the propagation of a disturbance in a material medium, without the transport of matter. It does not propagate in a vacuum (absence of matter). The disturbance corresponds to a variation in a mechanical property (speed, position, energy, etc.) of a material point.

An electromagnetic wave is a wave that consists of oscillating electric and magnetic fields, propagating through space without requiring a material medium. Unlike mechanical waves, electromagnetic waves can travel through a vacuum, as they do not rely on the movement of particles in a medium. Examples include light, radio waves, and X-rays.

I-2- Wave celerity

Wave celerity is a constant in a linear, homogeneous, isotropic and non-dispersive medium. It depends only on the nature of the medium.

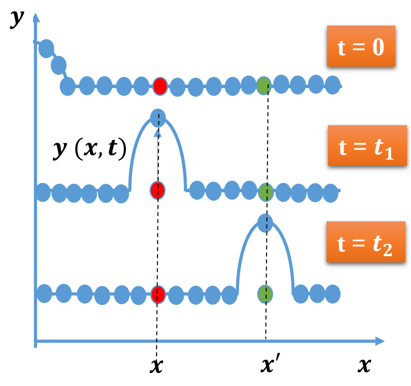
I-3- Wave vector

We define the direction of propagation of a wave in three-dimensional space by the wave vector $\vec{k}\,(k_x, k_y, k_z)$.

I-4- Propagation of a mechanical disturbance

I-4-1- Propagation of a transverse wave

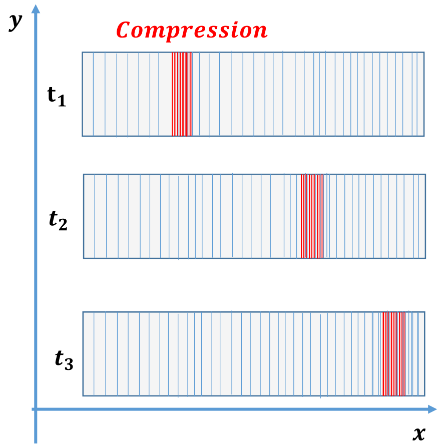
We consider a flexible string stretched between x=0 and infinity. At a point M with abscissa x on the string, we mark the particle with the color red. At time t=0, the string is subjected to a sudden vertical jerk at extremity x=0. This generates a disturbance (or tremor or vibration) which propagates along the string. At instant t= t_1, the disturbance arrives at point of abscissa x and at instant (t= t_2), it arrives at point of abscissa $\boldsymbol{x'}$ superior than x (figure 1).



The wave propagates along the Ox axis with a velocity V= x/t, while the particle (the matter) has vibrated along Oy, i.e. perpendicular to the direction of propagation; we say we have a transverse wave u(x,t) designates the position of the spot (point x) at time t.

I-4-2- Longitudinal wave propagation

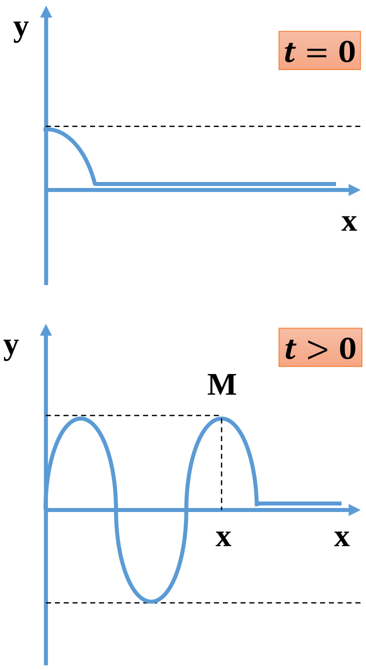
The string is now replaced by a long spring stretched horizontally. At the end of the spring, a few coils are compressed, then released suddenly. The disturbance created is the small displacement of each coil from its equilibrium position, which propagates along the tensioned spring. The disturbance propagated horizontally along the spring. During deformation, the points of the spring have moved horizontally from their equilibrium position. This wave is referred to as longitudinal.



I-4-3-Propagation of a sinusoidal wave in one direction with periodicity

A string is stretched along the x-axis from x = 0 to infinity. A sinusoidal vibration is imposed at x = 0, causing a disturbance that propagates along the string. The displacement of the string in the y direction at time (i.e., t = 0) at positin x is given by: $y(\mathrm{x}, 0)= U_0 \cos(kx)$

The first figure represents the shape of the wave at time t, when the string is initially disturbed. The second figure shows the wave's shape at any later time t.



The wave propagates with velocity v= x/Δt and reaches point M at abscissa x. Thus, the vibration at point M is the same as that at point O at position x-vt . We can therefore express it as:

$u_y(\mathrm{x}, \mathrm{t})= U_0 \cos(k(x\text{- vt}))$

This equation exhibits double periodicity: temporal periodicity (in time) and spatial periodicity.

Rewriting this equation: $y(\mathrm{x},\mathrm{t})= U_0 \cos(kx - \omega t)$, where $k = \frac{\omega}{V}$ is the wave number.

I-5- Double Periodicity

- Temporal periodicity: The function repeats after a time interval $y(\mathrm{x},\mathrm{t})$ given by:

$$\omega = \frac{2\pi}{T}$$

- Spatial periodicity: The function also repeats at intervals of the wavelength $y(\mathrm{x},\mathrm{t})$ given by:

$$\lambda = \mathrm{VT} = \frac{2\pi V}{\omega}$$

I-6- Measuring Period and Wavelength

- Period (T)**:** Fix a point on the string and measure the time for a full oscillation.
- Wavelength (λ)**:** Take a photo of the string and measure the distance between two consecutive points oscillating in phase.

I-7- The Wave Equation

A sinusoidal wave satisfies the wave equation:

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = 0$$

This is known as the d'Alembert wave equation.

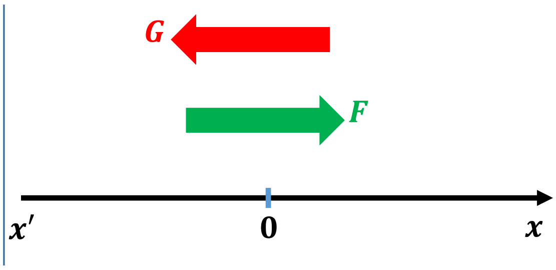
I-8- General Solution

Using d'Alembert's method, the general solution is:

$$y(x,t) = F(t-x/v) + G\,(t+x/v)$$

Where :

- **F(t-x/v):** corresponds to a wave propagating in the direction of increasing x (a progressive wave).
- **G (t+x/v):** corresponds to a wave propagating in the direction of decreasing x (a regressive wave)



Although and are sometimes labeled as incident and reflected waves, this is incorrect in general cases unless specific boundary conditions are imposed. This structured explanation should help students better understand the concept of sinusoidal progressive waves.

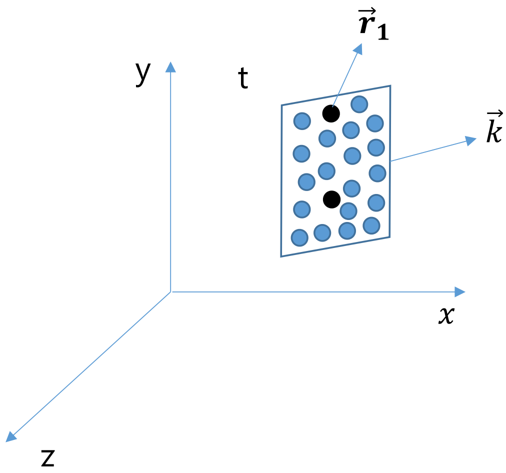
I-9- Plane wave and spherical wave

I-9-1- Plane wave

A plane wave is a wave in which all points in a plane perpendicular to the direction of propagation have the same phase, amplitude, and velocity at any given time. Mathematically, the plane wave is expressed in the following form:

$$y(x,t) = A\, e^{j(\omega t - \vec{k}\vec{r})}$$

- $y(\mathbf{r},t)$ is the wave function at position r and time t,
- A is the amplitude of the wave,
- k is the wave vector, which points in the direction of propagation and has a magnitude equal to the wavenumber k=$\frac{2\pi}{\lambda}$,
- r is the position vector,
- ω is the angular frequency of the wave,
- t is time,
- ϕ is the phase constant.



Key Characteristics of a Plane Wave:

1. **Wavefronts:** The wavefronts are infinite parallel planes perpendicular to the direction of propagation.
2. **Uniformity:** All particles or points on a given wavefront have the same displacement, velocity, and phase at any instant.
3. **Directionality:** The wave propagates in a single direction, specified by the wave vector k.
4. **Infinite Extent:** In theory, a plane wave extends infinitely in space, though in practice, it is an idealization used to simplify analysis.

Why is $y(x,t) = A\, e^{j(\omega t - \vec{k}\vec{r})}$ is the expression of a plane wave?

Step-by-Step Explanation

Step 1: Consider the Wave at Two Different Positions

We take the wave function at two different points r_1 and r_2:

$$y_1(r_1,t) = A\, e^{j(\omega t - \vec{k}\vec{r_1})}$$

$$y_2(r_2,t) = A\, e^{j(\omega t - \vec{k}\vec{r_2})}$$

If these two points belong to the same wavefront (i.e., they have the same wave amplitude and phase), then:

$$y_1(r_1,t) = y_2(r_2,t)$$

This simplies :

$$A\, e^{j(\omega t - \vec{k}\vec{r_1})} = A\, e^{j(\omega t - \vec{k}\vec{r_2})}$$

Since A and $e^{J\omega t}$are common on both sides, we can simplify to:

$$e^{-j(\vec{k}\vec{r_1})} = e^{-j(\vec{k}\vec{r_2})}$$

Taking the natural logarithm:

$$\vec{k}\vec{r_1} = \vec{k}\vec{r_2}$$

Which leads to :

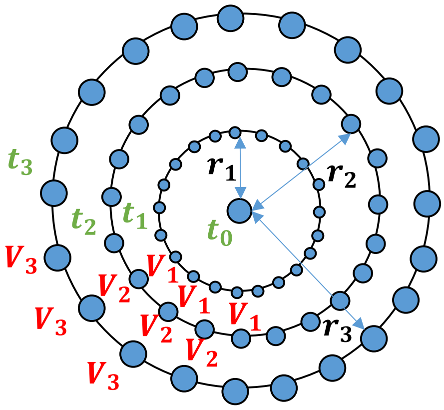
$$\vec{k}\,(\vec{r_1} - \vec{r_2})$$

is the equation of a plane in 3D space. Expanding this dot product,

$$k_x\,(x_2 - x_1) + k_y\,(y_2 - y_1) + k_z\,(z_2 - z_1) = 0 \Rightarrow \alpha x + \beta y + \gamma z = 0$$

I-9-2- Spherical wave

A spherical wave is a wave in which points having the same vibratory state form concentric spheres, with the center of the spheres being the source of the waves. In an isotropic and homogeneous medium, the wave equation is expressed in spherical coordinates



$$\frac{\partial^2 u}{\partial r^2} - \frac{2}{r}\frac{\partial u}{\partial r} - \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = 0$$

In the case of a sinusoidal wave, the vibration of a point M located at a distance r from the source is:

$$U(r,t) = \frac{u_0}{r} \quad \cos[(\omega(t\text{- r/v})]$$

I-10- Phase velocity

When a wave propagates through a given medium, the speed of propagation is typically independent of frequency, except in dispersive media, where it can vary with frequency.

For example:

The velocity of sound in air is around 340 m/s, regardless of the frequency of the sound being propagated. We then have the relationship $V_\varphi = \frac{\omega}{k}$, in which case we say that the medium is non-dispersive: the phase velocity is equal to the wave velocity C.

I-11- Group velocity

In many physical systems, the propagation of waves depends on their wavelength. A medium that causes waves of different wavelengths to travel at different velocity is called a dispersive medium. To illustrate the concept of group velocity, we consider the superposition of two sinusoidal waves, u_1 and u_2 , with slightly different frequencies but identical amplitudes.

It may seem that a medium separates waves of different wavelengths; such a medium is called dispersive. We consider two sine waves, u_1 and u_2, with the same amplitude but slightly different frequencies being propagated.

- Let the central angular frequency be:

$$\omega_0 = \frac{\omega_1+\omega_2}{2} \text{ and } \Delta\omega = \omega_2 - \omega_1$$

which gives

$$\omega_1 = \omega_0 - \frac{\Delta\omega}{2} \text{ and } \omega_2 = \omega_0 + \frac{\Delta\omega}{2}$$

- Similarly, for the wave numbers:

$$K_1 = k_0 - \Delta k \text{ and } K_2 = k_0 + \Delta k$$

Thus, the two wave equations are:

$$u_1(x,t) = u_0\, e^{i((\omega_0-\frac{\Delta\omega}{2})\mathrm{t}-\,(k_0-\Delta k)x))}$$

$$u_2(x,t) = u_0\, e^{i((\omega_0+\frac{\Delta\omega}{2})\mathrm{t}-\,(k_0+\Delta k)x))}$$

Using Euler's formula, the sum of these two waves results in:

$$u(x,t) = 2\,u_0\, e^{i(\frac{\Delta\omega}{2}\mathrm{t}-\frac{\Delta k}{2}x)}\; e^{i(\omega_0\,\mathrm{t}-\,k_0 x)}$$

Here, we identify two components:

- Envelope function (modulating wave):

$$A(\omega) = 2\,u_0\, e^{i(\frac{\Delta\omega}{2}\mathrm{t}-\frac{\Delta k}{2}x)}$$

This wave propagates with the group velocity:

$$V_g = \frac{\Delta\omega}{\Delta k} = \frac{d\omega}{dk}$$

which represents the velocity at which the energy or information is transmitted.

- **Carrier wave**

$$B(\omega) = e^{i(\omega_0 t - k_0 x)}$$

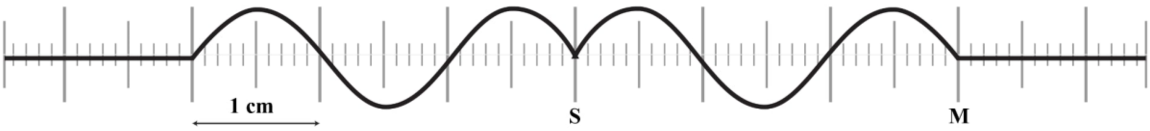
This wave propagates at the phase velocity:

$$V_\varphi = \frac{\omega_0}{k_0}$$

Exercises and solved problems

Exercise 1:

At time t=0, a progressive sinusoidal mechanical wave of frequency f = 50 Hz is created at a point S on the water surface. The figure below illustrates a vertical section of the water surface at an arbitrary time t. A graduated ruler is included to indicate the scale.



1- The wavelength is:

$* \lambda = 0.2\ cm$ $* \lambda = 4\ cm$ $* \lambda = 2\ cm$ $* \lambda = 6\ cm$

2- The propagation velocity of the wave at the water surface is:

$* V = 2\frac{m}{s}$ $* V = 1\frac{m}{s}$ $* V = \frac{3m}{s}$ $* V = 8.10^{-4}$ m/s

3- The time t, corresponding to the moment when the cross-section of the water surface is shown:

$* t = 8\ s$ $* t = 0.03\ s$ $* t = 0.3\ s$ $* t = 3$ s

4- Consider a point M located on the surface of the water, at a distance SM=6 cm from the wave source S. Point M oscillates in the same way as S, but its motion is delayed by a time τ due to the wave propagation. The relationship between the displacement (elongation) of point M and the displacement of the source S, is given by the equation:

$$* y_M(t) = y_S(t - 0.3) \quad * y_M(t) = y_S(t + 0.3)$$

Solution:

1) $\lambda = 2$cm

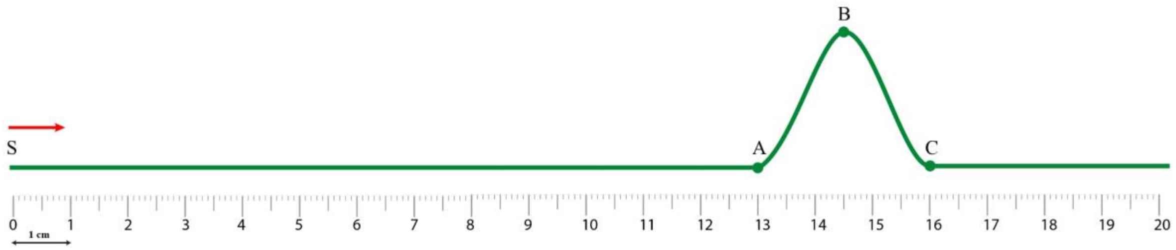
2) V=$\frac{\lambda}{T} = \lambda f = 0.02 \times 50 = 1\ m/s$

3) V = $\frac{SM}{\Delta t} = \frac{SM}{t_2 - t_1} \Rightarrow t_2 - t_1 = t = \frac{SM}{V} = 0.03\ s$

4) $y_M(t) = y_S(t - 0.3)$

Exercise 2:

The figure below illustrates the propagation of a wave along a string. It depicts the shape of the string at the specific time t=40 ms. The wave originated from a source at time $t_0 = 0$



1- Defining a progressive mechanical wave.

2- What is the nature of the wave? What is its dimension?

3- Determine the motion of each point based on whether it's moving toward a crest or trough.

4- Calculate V the velocity of wave propagation along the string.

5- At what point does point C stop (position at which propagation begins)?

6- Show graphically the appearance of the string at time t= 10ms.

Solution:

1) A progressive wave is a wave that propagates in the direction of increasing x or of positive x.

2) We have a transverse wave, when the wave propagates in the direction of x and the particle vibrates in y direction, which means that the direction of the wave propagation is perpendicular the particle displacement. It is a one dimension wave.

3) Point A remains stationary while, the point B moves downward and point C upward.

4) $V = \frac{AC}{\Delta t} = 4\ m/s$

5) For point C to stop, the disturbance must exceed the distance AC:

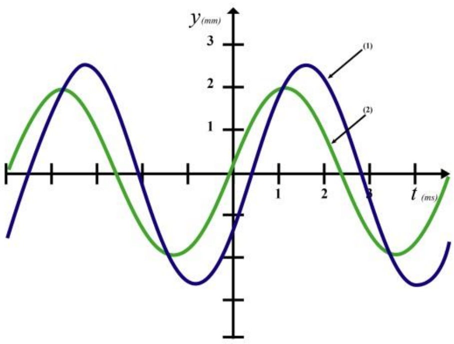
$V = \frac{AC}{\Delta t} \Rightarrow V = \frac{AC}{t_1 - t_0} \Rightarrow (t_1 - t_0)V = AC \Rightarrow t_1 = t_0 + \frac{AC}{V} = 47.5\ ms$

6) $= \frac{x}{t_1 - t_0} \Rightarrow x = 4\ cm$

Exercise 3:

Two microphones, M_1 and M_2, are placed near the axis perpendicular to the speaker's membrane and passing through its center O (see Document 1). The speaker is connected to a sinusoidal voltage generator with an adjustable frequency. The microphones are connected to an oscilloscope, whose settings are provided in the table below

Document 2 is a reproduction of the oscillogram obtained.



Data: Under the conditions of the experiment, the speed of sound in air is approximately 340 m/s.

1. Determine the temporal period T and the frequency f of the sound wave emitted by the speaker.
2. Deduce the spatial period λ (wavelength) of this wave.
3. These curves are obtained for a minimal distance d_{min} between the two microphones:

    3.1. Determine the time delay between the two microphones.

    3.2. Deduce the minimal distance d_{min} separating the two microphones.

    3.3. For what other distances separating the two microphones would the same oscillogram be obtained?

    3.4. Microphone M_2 is moved closer to M_1 by a distance equal to λ/2.

Solution:

1) T = 5 ms, $f = \frac{1}{T} = 200\ Hz.$

2) $V = \frac{\lambda}{T} \Rightarrow \lambda = VT = 1.7\ m$

3.1. $\tau = \frac{1\ 10^{-3}}{2} = 0.5\ ms$

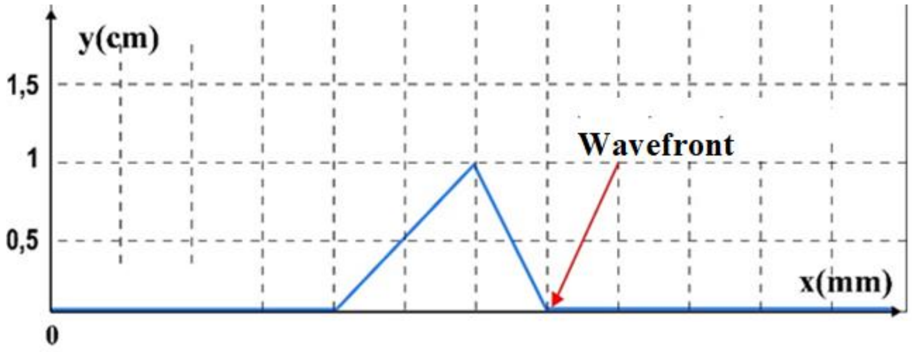
3.2. $V = \frac{d_{min}}{\tau} \Rightarrow d_{min} = V\tau = 17\ cm$

3.3. $d = d_{min} + \lambda$

3.4. $d = d_{min} + \frac{\lambda}{2}$

Exercise 4:

A wave propagates along a string along the x'Ox axis (see figure). At time t=0, the wavefront (the leading edge of the disturbance) is at x=0. A photograph of the string is taken at time t_1=4 s, and the resulting image is shown in the figure below.



1- What is the nature of the wave (transverse or longitudinal)?

2- Calculate the speed of the wave along the string.

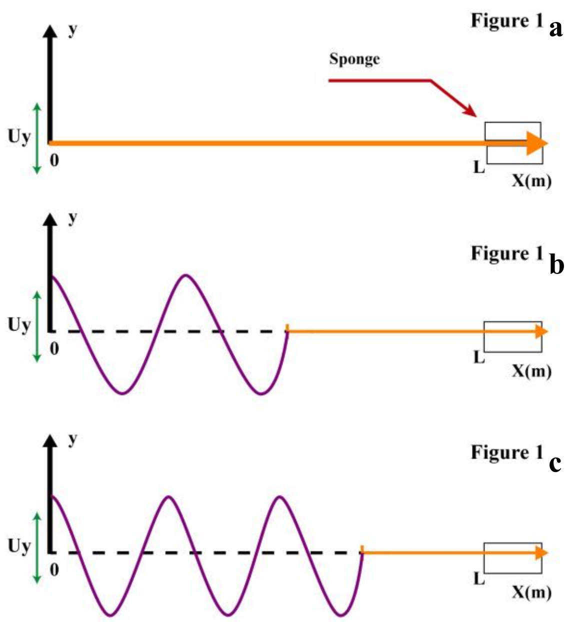
3- What is the duration τ of the motion of a point on the string as the wave passes?

4- Draw, on the given graph below, the appearance of the string at the time t_2=5 s.

5- Describe the motion of point M, located at x= 5 mm.

Exercise 5:

We consider a flexible string stretched along the x'Ox axis between x=0 and x=L. The string is clamped between two absorbent sponges. The length of the string is L=1 m (see Figure a). The end at x=0 is subjected to a sinusoidal vibration $y\ (t) = A\ \cos(\omega t + \varphi)$ along the Oy axis. The vibration amplitude is A=3 mm. The wavelength $\lambda = 0.4\ m$.



Figures b and c show two photographs of the string at two moments, t_1 and t_2 , such that:

$$t2 - t1 = 0.02s.$$

1- What is the purpose of clamping the string between two sponges $x = L$?

2- What is the nature of this wave?

3- What is the velocity of the wave? Deduce the frequency f of the source.

4- Determine the instants t_1 et t_2.

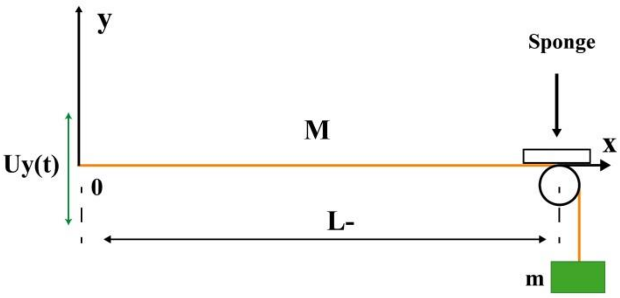
5- What is the equation of motion of a point M with abscissa x on the string? Deduce the value of φ?

6- What time is the end x=L of the string reached?

Exercise 6:

A flexible string is stretched along the x'Ox axis. At x=L, the string passes through the groove of a pulley, and at its end, a mass m is attached to tension it. An absorbent sponge is pressed against the pulley to prevent any reflection. The end at x=0 of the string is subjected to a sinusoidal transverse vibration along the Oy axis (see figure).

$u_y(t) = U_0 \cos(120\pi t)$, Avec $U_0 = 3.10^{-3}m$ and t is expressed in seconds. Given: L = 5m and $\lambda = 1m$.



1- Give the expression $u_y(x,t)$ or the vibration of point M at abscissa x.

2- What is the position of the points that vibrate:

a- In phase with O

b- Out of phase relative to O

c- Phase-shifted by $\frac{\pi}{2}$ relative to O

3- Plot on the same graph the displacement of point O (located at x = 0) and that of point A where OA = 2.75 m.

4- Represent the appearance of the string at times $t_1 = 0.04$ s and $t_2 = 0.05$ s.

Chapter II: The vibrating string

I-1- Introduction to the Vibrating String

In this chapter, we will explore the fundamental principles governing vibrating strings. We will begin by deriving the wave equation using Newtonian mechanics and examining its physical significance. Then, we will discuss d'Alembert's solution and its implications for wave propagation. Following this, we will analyze harmonic progressive plane waves, particle velocity, and force at a point. We will also introduce the concept of mechanical impedance and investigate wave reflection and transmission in different cases:

- The case of two semi-infinite strings
- A semi-infinite string terminated by a terminal impedance at
- A semi-infinite string terminated by a terminal impedance at

Next, we will study standing waves and the case where a semi-infinite string is terminated at by a terminal impedance such that, leading to quasi-stationary waves. Finally, we will explore free and forced oscillations of a string of length, including:

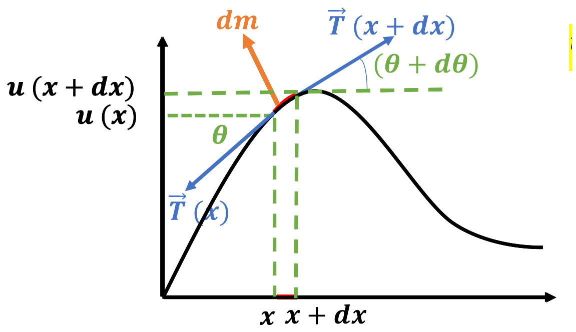
- Free oscillations of a string fixed at both ends
- Forced oscillations
- Power and energy of waves

II-2- The propagation equation of the wave

We consider a semi-infinite string under tension T, stretched along the $x'Ox$ axis, extending from x=0 to x=∞. The linear mass density of the string is denoted by μ\muμ, defined as:

$$\mu = \frac{dm}{dx}$$

which represents the mass per unit length of the string.



For simplicity, we assume that the weight of the string is negligible compared to the tension T.

At time t=0, a disturbance is introduced at x=0, and at a later time t, the wave reaches point x.

To analyze the motion of the wave, we focus on the infinitesimal segment of the string between x and x+dx, where dx represents a small displacement along the string

Applying Newton's Second Law:

- **Along the Ox axis:**

$$T(x+dx)-T(x) = dm.\ddot{u}_x = 0$$

Since the tension is assumed to be constant along the string, there is no acceleration in the x direction.

- **Along the Oy axis:**

Here, we assume the vertical displacement of the string is $u_y(x,t)$= u(x,t)

$$T(x+dx)\sin(\theta + d\theta) - T(x)\sin\theta = dm\,\frac{\partial^2 u}{\partial t^2}$$

Since the tension T is constant, we approximate the sine function using the small-angle approximation:

$$sin(\theta) = \tan(\theta) = \left.\frac{du}{dx}\right|_x = \left.\frac{\partial u}{\partial x}\right|_x$$

$$sin(\theta + \Delta\theta) = \tan(\theta + \Delta\theta) = \left.\frac{du}{dx}\right|_{x+\Delta x} = \left.\frac{\partial u}{\partial x}\right|_{x+\Delta x}$$

Thus, expanding using a Taylor series, we get:

$$T\left[\left.\frac{\partial u}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial u}{\partial x}\right|_x\right] = dm\,\frac{\partial^2 u}{\partial t^2}$$

Dividing by dx and using the definition of linear mass density $\mu = \frac{dm}{dx}$, we obtain the wave equation**:**

$$T\frac{\partial^2 u}{\partial x^2} = \mu\,\frac{\partial^2 u}{\partial t^2}$$

Rearranging, we get:

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = 0$$

where the wave velocity is given by:

$$V = \sqrt{\frac{T}{\mu}}$$

This is known as D'Alembert's equation, or the wave propagation equation.

II-3- Solution to the propagation equation

The general solution to the wave equation can be expressed as:

$$u(x,t) = F\left(t - \frac{x}{V}\right) + G\left(t + \frac{x}{V}\right)$$

where:

- $F\left(t - \frac{x}{V}\right)$ represents a wave propagating in the positive x-direction**.**
- $G\left(t + \frac{x}{V}\right)$ represents a wave propagating in the negative x-direction.

II-4- Solution of the harmonic progressive plane wave

For a harmonic progressive wave, the displacement is expressed as:

$$u_y(x,t) = u_0 e^{j(\omega t - kx)}$$

Where:

- u_0 is the amplitude of the wave,
- ω is the angular frequency,
- $k = \frac{\omega}{V}$ is the wave number, which represents the magnitude of the wave vector.

II-5- Particle velocity:

The particle velocity is obtained by differentiating $u_y(x,t)$ with respect to time:

$$\dot{u}_y = \frac{\partial u}{\partial t} = j\omega u_y = j\omega u_0 e^{j(\omega t - kx)}$$

II-6- Force at a point:

The force at a point refers to the projection of the force exerted on the right part of the string by the left part, along the y axis:

$$F(x,t) = -T_0 \frac{\partial u}{\partial x}$$

Since $u_y(x,t) = u_0 e^{j(\omega t - kx)}$, differentiating with respect to x gives:

$$F(x,t) = -T_0\left(-jku_0 e^{j(\omega t - kx)}\right) = T_0 jku_0 e^{j(\omega t - kx)})$$

II-7- Mechanical impedance

The mechanical impedance at a point is defined as the ratio of the force to velocity at that point:

$$Z(x) = \frac{F(x,t)}{\dot{u}_y}$$

Substituting the expressions for F(x,t) and $\dot{u}_y$:

$$Z(x) = \frac{T_0 jku_0 e^{j(\omega t-kx)})}{j\omega u_0 e^{j(\omega t-kx)}}$$

Simplifying:

$$Z(x) = \frac{kT_0}{\omega}$$

Since k = $\frac{\omega}{V}$, we get:

$$Z(x) = \frac{T_0}{V} = \sqrt{\mu T_0} \quad \forall\, x$$

This quantity is known as the characteristic impedance:

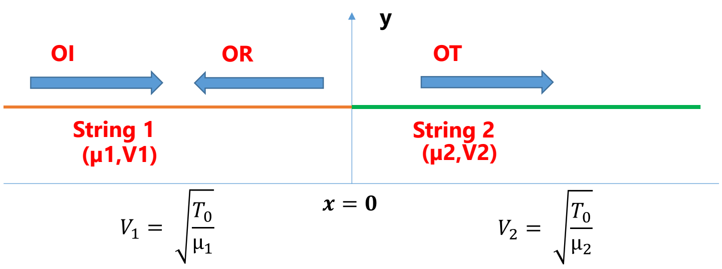
$$Z_c = \sqrt{\mu T_0}$$

II-8- Reflection and transmission phenomenon of waves

II-8-1- Case of two semi-infinite strings

We consider two semi-infinite strings under the same tension *T*, stretched along the *x'Ox* axis. These two strings are joined at *x=0*, forming an interface between two different media.

- The first string (medium 1) occupies the region x<0 and has a linear mass density $\mu 1$.
- The second string (medium 2) occupies the region x>0 and has a linear mass density $\mu 2$.



A sinusoidal traveling wave is sent from $-\infty$ through string 1. When it reaches x=0, the wave undergoes partial reflection and partial transmission:

- Incident wave (traveling toward x=0):

$$u_i(x,t) = U_i\ e^{j(\omega t - k_1 x)}$$

- Reflected wave (traveling back in medium 1):

$$u_r(x,t) = U_r\ e^{j(\omega t + k_1 x)})$$

- Transmitted wave (traveling forward in medium 2):

$$u_t(x,t) = U_t\ e^{j(\omega t - k_2 x)}$$

Wave Equation in Both Regions

In the region $x<0$ (medium 1), the total displacement results from the superposition of the incident and reflected waves**:**

$$u_1(x,t) = U_i\ e^{j(\omega t - k_1 x)} + U_r\ e^{j(\omega t + k_1 x)}$$

In the region $x > 0$ (medium 2), only the transmitted wave propagates:

$$u_2(x,t) = U_t\ e^{j(\omega t - k_2 x)}$$

Reflection and Transmission Coefficients

To determine the reflection coefficient R and the transmission coefficient T, we apply two physical conditions at **x=0**:

$$R = \frac{u_r}{u_i} \quad \text{and } T = \frac{u_t}{u_i}$$

1. Continuity of Displacement at x=0

The wave displacement must be continuous at the junction:

$$u_1(0,t) = u_2(0,t)$$

$$U_i\ e^{j\omega(t)} + U_r\ e^{j\omega(t)} = U_t\ e^{j\omega(t)}$$

Dividing by $e^{j\omega(t)}$, we obtain:

$$1+R = T \quad .....(1)$$

2. Force Equilibrium at x=0 (Fundamental Relation of Dynamics)

Newton's second law applied at the junction gives:

$$[F_y \text{ from the left end of the string}] + [F_y \text{ from the right end of string}] = m\ddot{u}$$

Since the force on the string is given by F(x,t) = $-T_0 \frac{\partial u}{\partial x}$, we get:

-T $\frac{\partial u_1}{\partial x}$ + T $\frac{\partial u_2}{\partial x}$ = 0 ⇒ -T [-j $k_1 U_i\ e^{j(\omega t-kx)}$ + j$k_1 U_r e^{j(\omega t+kx)}$]$_{x=0}$ + T [-j$k_2 U_t e^{j(\omega t-kx)}$]$_{x=0}$ = 0

Simplifying:

$$Z_{c1}\ (U_i - U_r\ ) = Z_{c2}\ U_t$$

where Z(x) =$\frac{T_0}{V} = \sqrt{\mu T_0}$ is the characteristic impedance of the string. This gives the second equation:

$$1\text{-}R = \frac{Z_{c2}}{Z_{c1}}\,T\text{.....(2)}$$

3. Final Expressions for *R* and *T*

Solving Equations (1) and (2) simultaneously:

- Reflection coefficient:

$$R = \frac{Z_{c1} - Z_{c2}}{Z_{c1}+Z_{c2}} = \frac{\sqrt{\mu_1} - \sqrt{\mu_2}}{\sqrt{\mu_1}+\sqrt{\mu_2}}$$

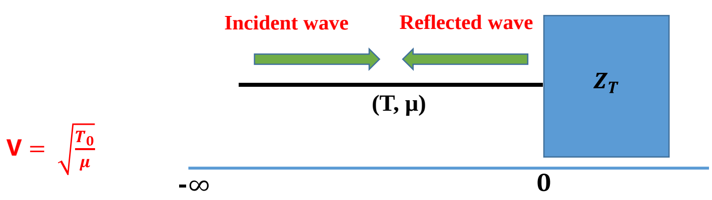
- **Transmission** coefficient:

$$T = \frac{2Z_{c1}}{Z_{c1}+Z_{c2}} = \frac{2\sqrt{\mu_1}}{\sqrt{\mu_1}+\sqrt{\mu_2}}$$

II-8-2-The case of a semi-infinite string terminated by a terminal impedance Z_L at x=0

We consider a semi-infinite string under tension T, aligned along the x axis, extending from x= $-\infty$ to x=0. The string terminates at x=0 with a load impedance .

A sinusoidal incident wave with angular frequency ω and amplitude Ui propagates along the string from x=−∞. At x=0, the wave undergoes partial reflection due to the impedance mismatch.



Mathematical Formulation

Dr. Badis RIAH

- Incident wave (traveling toward x=0):

$$u_i(x,t) = U_i\ e^{j(\omega t - k_1 x)}$$

- Reflected wave (traveling back in medium 1):

$$u_r(x,t) = U_r\ e^{j(\omega t + kx)}$$

- The total wave displacement in the string is given by:

$$u(x,t) = U_i\ e^{j(\omega t - kx)} + U_r\ e^{j(\omega t + kx)}$$

where $k = \frac{\omega}{V}$is the wave number, and $V = \sqrt{\frac{T}{\mu}}$ is the wave velocity.

- **Impedance of the String at *x***

The mechanical impedance at a point *x* in the string is defined as:

$$Z(x) = \frac{F(x,t)}{\dot{u}_y}$$

Substituting the wave expressions:

$$Z(x) = -T\ \frac{[-j\frac{\omega}{V}U_i\ e^{j\left(\omega t-\frac{x}{V}\right)}+j\frac{\omega}{V}U_r\ e^{j\left(\omega t+\frac{x}{V}\right)}]}{j\omega\ U_i\ e^{j\left(\omega t-\frac{x}{V}\right)}+j\omega\ U_r\ e^{j\left(\omega t+\frac{x}{V}\right)}}$$

Simplifying:

$$Z(x) = \frac{T}{V}\frac{[U_i\ e^{j\omega\frac{x}{v}} - U_r\ e^{j\omega\frac{x}{v}}]}{U_i\ e^{j\omega\frac{x}{v}} + U_r\ e^{j\omega\frac{x}{v}}}$$

As x→0, we set Z(x) equal to the terminal impedance :

$$Z(x=0) = Z_T = \frac{T}{V}\frac{1-R}{1+R}$$

where $R = \frac{u_r}{u_i}$ is the reflection coefficient.

- **Expression for the Reflection Coefficient**

Solving for R:

$$R = \frac{Z_c - Z_T}{Z_c + Z_T}$$

where $z_c = \frac{T_0}{V}$ is the characteristic impedance of the string.

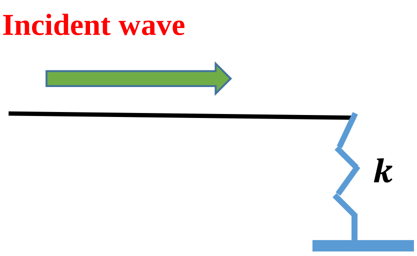
II-8-3- Special Cases of Reflection at a Terminated String

The reflection coefficient R depends on the nature of the termination impedance Z_T. Let's explore three special cases:

1. Case 1: The String is Terminated by a Spring ($Z_T = \frac{k}{j\omega}$)

1.1. Physical interpretation:

- The end of the string is attached to a spring with a spring constant k.
- The spring resists displacement but does not completely block motion.
- The restoring force exerted by the spring depends on its stiffness and the wave frequency.



1.2. Reflection coefficient:

$$R = \frac{Z_{c1} - \frac{k}{j\omega}}{Z_{c1} + \frac{k}{j\omega}}$$

- Magnitude of reflection: |R|=1 ⟶ Total reflection (no energy is lost, but the phase may change).
- **Phase shift:**

$$\theta = \arg(R) = 2\arctan\left(\frac{K}{Z_c\omega}\right)$$

 If k is small, the wave is almost fully reflected with little phase shift.

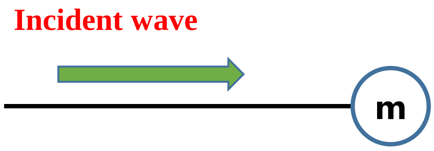
 If k is large, the phase shift approaches π (a phase inversion occurs).

2. Case 2: The String is Terminated by a Mass ($Z_T = jm\omega$)

1.1. Physical explication :

- The end of the string is attached to a small mass *m*.

- The mass resists changes in motion due to inertia.
- The effect of the mass increases with frequency ω, meaning higher-frequency waves experience a stronger effect.



1.2. Reflection coefficient:

$$R = \frac{Z_{c1} - jm\omega}{Z_{c1} + jm\omega}$$

- Magnitude of reflection: |R|=1 ⟶ Total reflection (no energy is lost, but the phase may change).
- **Phase shift:**

$$\theta = \arg(R) = -\,2\arctan\frac{m\omega}{Z_c}$$

□ For low-frequency waves, the mass has little effect, and the phase shift is small.

□ or high-frequency waves, the mass behaves like a fixed end, causing a phase shift approaching π (inverted wave reflection).

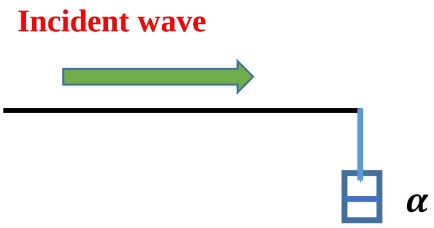
3. Case 3: The String is Terminated by a Damper ($Z_T = \alpha$)

1.1. Physical interpretation:

- The end of the string is connected to a damping system (like a shock absorber or a dashpot).
- The damper dissipates wave energy into heat or other forms of energy.
- Depending on the value of α\alpha$\alpha$, the wave may be partially absorbed or partially reflected.

1.2. Reflection coefficient:

$$R = \frac{Z_{c1} - \alpha}{Z_{c1} + \alpha}$$



- Magnitude of reflection: *|R|<1 for α>0* → Some energy is absorbed.

- **Phase shift**

$\theta = 0\ if\ Z_{c1} > \alpha,\ \theta = \pi\ si\ Z_{c1} < \alpha$

□ If the characteristic impedance Z_c is larger than α, the reflected wave has no phase shift.

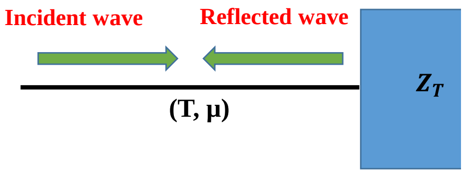
□ If Z_c is smaller than α, the reflected wave is inverted (phase shift of π).

II-8-4- The case of a semi-infinite string terminated by a terminal impedance Z_T at x = L

We consider a semi-infinite string stretched with tension T along the x′Ox axis, extending from −∞ to x=L. The string is terminated at x=L with an impedance .

A sinusoidal wave is generated from x=−∞ and propagates towards x=L. Upon reaching the termination, the wave undergoes reflection. Our goal is to determine the reflection coefficient: $R = \frac{U_r}{U_i}$.

$V = \sqrt{\frac{T_0}{\mu}}$



- **Wave Representation**

The incident and reflected waves can be expressed as:

$$u_i(x,t) = U_i\ e^{j(\omega t - k_1 x)}$$

$$u_r(x,t) = U_r\ e^{j(\omega t + kx)}$$

The total wave displacement in the string is given by:

$$u(x,t) = U_i\ e^{j(\omega t - kx)} + U_r\ e^{j(\omega t + kx)}$$

At x=L, the reflected wave satisfies:

$$u_r(L,t) = R\,u_i\,(L,t) \Longrightarrow U_r\ e^{j(\omega t + kL)} = RU_i\ e^{j(\omega t - kL)} = RU_i\ e^{-2jKL}$$

Thus, the total displacement at any point is:

$$u(x,t) = U_i\left[e^{j(\omega t-kx)} + Re^{j(\omega t+kx-2kl)}\right]$$

- **Impedance Calculation**

The impedance at any point *x* in the string is given by:

$$Z(x) = \frac{F(x,t)}{\dot{u}_y}$$

Substituting the expression for *u(x,t)*:

$$Z(x) = Z_c\ \frac{e^{jk(L-x)} - \ R\,e^{-jk(L-x)}}{e^{jk(L-x)} + \ Re^{-jk(L-x)}}$$

Since the reflection coefficient at x=L is:

$$R = \frac{Z_c - Z_T}{Z_c + Z_T}$$

we substitute this into the impedance equation:

$$Z(x) = -\,T\,\frac{U_i\left[-j\,\frac{\omega}{V}\,e^{j(\omega t-kx)}+j\,\frac{\omega}{V}\,R\,e^{j(\omega t+kx-2kL)}\right]}{U_i\left[j\,\omega\,e^{j(\omega t-kx)}+j\,\omega\,R\,e^{j\omega(t+kx-2kL)}\right]}$$

$$Z(x) = Z_c\ \frac{e^{-jkx} - R\,e^{j(kx-2kL)}}{e^{-jkx} + R\,e^{j(kx-2kL)}}\,\frac{e^{jkL}}{e^{jkL}}$$

$$Z(x) = Z_c\ \frac{e^{jk(L-x)} - \ R\,e^{-jk(L-x)}}{e^{jk(L-x)} + \ Re^{-jk(L-x)}}$$

$$Z(x) = Z_c\ \frac{e^{jk(L-x)} - \ \dfrac{Z_c - Z_T}{Z_c + Z_T}\ e^{-jk(L-x)}}{e^{jk(L-x)} + \dfrac{Z_c - Z_T}{Z_c + Z_T}\,e^{-jk(L-x)}}$$

$$Z(x) = Z_c\ \frac{Z_c\,[e^{jk(L-x)} - e^{-jk(L-x)}] + Z_L\,[e^{jk(L-x)} + e^{-jk(L-x)}]}{Z_c\,[e^{jk(L-x)} + e^{-jk(L-x)}] + Z_L\,[e^{jk(L-x)} - e^{-jk(L-x)}]}$$

$$Z(x) = \ Z_c\ \frac{2j\,Z_c \sin[k(L-x)] + 2\,Z_L \cos[k(L-x)]}{2\,Z_c \cos[k(L-x)] + 2j\,Z_T \sin[k(L-x)]}$$

We devide on $\cos[k(L-x)]$, we obtain:

$$Z(x) = Z_c\ \frac{Z_T + j\,Z_c \tan[k(L-x)]}{Z_c + j\,Z_T \tan[k(L-x)]}$$

- **Physical Interpretation**

  - The impedance at any position x depends on the termination impedance Z_T and the distance from the termination**.**
  - This equation shows how the wave impedance varies along the string, oscillating due to reflections.
  - If $Z_T \to \infty$ (a fixed end), then $R=1$, leading to total reflection.
  - If $Z_T = Z_c$, then $R=0$, meaning no reflection occurs (matched impedance).

II-9- Standing Waves on a Semi-Infinite String

II-9-1- Case 1: String Terminated at *x=0* with */R/=1*

1. Understanding Standing Waves

A standing wave forms when a wave reflects back on itself with full reflection (|R|=1). This happens when the terminal impedance fully reflects the wave without absorbing any energy.

The total wave on the string consists of:

- An incident wave traveling toward x=0.
- A reflected wave traveling in the opposite direction.

$$U(x,t) = U_i\ e^{j(\omega t-kx)} + U_r\ e^{j(\omega t+kx)}$$

Since $= 1.\ e^{j\theta}$, we substitute $u_r = R\, u_i = u_i\ e^{j\theta}$

$$U(x,t) = U_i\ e^{j(\omega t-kx)} + U_i\ e^{j(\omega t+kx+\theta)}$$

Rearranging:

$$U(x,t) = U_i\ e^{j\left(\omega t+\frac{\theta}{2}\right)}\ [\ e^{-j\left(kx+\frac{\theta}{2}\right)} + e^{+j\left(kx+\frac{\theta}{2}\right)}]$$

Taking the real part:

$$U(x,t) = 2\ U_i \cos(\omega t+\frac{\theta}{2})\ \cos(kx+\frac{\theta}{2})$$

2. Interpretation of the Equation

This equation is of the form:

$$u(x,t) = F(t)\ g(x)$$

This is the mathematical form of a standing wave, where:

- F(t) = $\cos(\omega t + \frac{\theta}{2})$ controls the oscillation in time.
- g(x) = $2\ U_i \cos(kx + \frac{\theta}{2})$ determines the variation of amplitude along x.

3. Nodes and Antinodes

- **Nodes:** Points where the string does not move.
- **Antinodes:** Points where the amplitude is maximum.

Antinode positions (maximum amplitude) :

$$\cos\left(kx + \frac{\theta}{2}\right) = \ +1$$

Solving for x :

$$\omega\frac{x}{V} + \frac{\theta}{2} = 2n\pi \Longrightarrow \ x_{Antinodes} = n\lambda - \ \frac{\lambda\theta}{4\pi}$$

And:

$$\cos\left(kx + \frac{\theta}{2}\right) = -1$$

Solving for x :

$$kx + \frac{\theta}{2} = (2n + 1)\pi$$

$$x_{Antinodes} = \frac{(2n + 1)\lambda}{2} - \ \frac{\lambda\theta}{4\pi}$$

Node positions (no movement)

$$\cos\left(\omega\frac{x}{V} + \frac{\theta}{2}\right) = 0$$

Solving for x :

$$\omega\frac{x}{V} + \frac{\theta}{2} = \frac{(2n + 1)\pi}{2}$$

$$x_{Nodes} = (2n + 1)\frac{\lambda}{4} - \ \frac{\lambda\theta}{4\pi}$$

4. Observations

- The distance between two successive antinodes (or nodes) is λ/2.
- The distance between a node and the nearest antinode is λ/4.

II-9-2- Case 2: String Terminated at *x=L* with */R/=1*

Now, we analyze a semi-infinite string that is terminated at x=L instead of x=0.

The total wave is:

$$u(x,t) = U_i\ e^{j(\omega t-kx)} + U_r\ e^{j(\omega t+kx)}$$

At x=L:

$$u_r(L,t) = R\ u_i\ (L,t)$$

$$U_r\ e^{j(\omega t+kL)} = RU_i\ e^{j(\omega t-kL)}$$

$$U_r = RU_i\ e^{-2jKL}$$

Since |R|=1, we write R= $e^{j\theta}$, giving:

$$u(x,t) = U_i\left[e^{j(\omega t-kx)} + Re^{j(\omega t+kx-2kl)}\right]$$

Rearranging:

$$u(x,t) = 2U_i \cos\left(\omega t - kL + \frac{\theta}{2}\right)\cos\left(k(x-L) + \frac{\theta}{2}\right)$$

- **Nodes and Antinodes**

Antinode positions (maximum amplitude) :

$$\cos\left(k(x-L) - \frac{\theta}{2}\right) = \pm 1$$

Solving for x:

$$k(L-x) - \frac{\theta}{2} = 2n\pi$$

$$x_{Antinodes} = L - \frac{\theta\lambda}{4\pi} + n\lambda$$

Node positions (minimum amplitude) :

$$\cos\left(k(x-L)-\frac{\theta}{2}\right)=0$$

Solving for x:

$$x_{Nodes}=L-\frac{\theta\lambda}{4\pi}+\frac{(2n+1)\lambda}{4}$$

- **Key Takeaways**

- Standing waves form when the reflection coefficient satisfies $|R|=1$.
- The amplitude varies along the string, creating nodes and antinodes.
- The wavelength λ determines the spacing of nodes and antinodes:
  - Node-to-node distance is $\lambda/2$.
  - Node-to-antinode distance is $\lambda/4$.
- The reflection phase θ shifts the positions of nodes and antinodes.

II-10- Standing Wave Ratio (Quasi-Stationary Wave)

II-10-1- Partial Reflection and the Reflection Coefficient

In general, when a wave encounters an impedance mismatch, part of the wave is reflected and part is transmitted. The reflection coefficient is given by:

$$R=|R|\,e^{j\theta}$$

Where:

- $|R|$ is the magnitude of the reflection coefficient, always satisfying $0\leq|R|<1$.
- θ is the phase shift introduced by the reflection.

Since $|R|<1$, the wave is not purely standing but a combination of traveling and standing wave components, leading to a quasi-stationary wave**.**

II-10-2-Resulting Wave Equation

The total wave is the sum of the incident and reflected waves:

$$u(x,t)=U_i[e^{j\left(\omega t-\frac{x}{v}\right)}+|R|\,e^{j\left(\omega t+\frac{x}{v}\right)}]$$

Factoring out $U_i[e^{j\left(\omega t-\frac{x}{v}\right)}$:

$$u(x,t) = \left(1 + |R|\, e^{j\left(2\omega\frac{x}{v}+\theta\right)}\right)\ e^{j\left(\omega t-\frac{x}{v}\right)}$$

This represents a progressive wave whose amplitude varies with x, given by:

$$U(x) = U_i \left|1 + |R|\, e^{j\left(2\omega\frac{x}{v}+\theta\right)}\right|$$

Using the identify $\left|a + be^{j\phi}\right| = \sqrt{a^2 + b^2 + 2ab\cos\phi}$, we obtain:

$$U(x) = U_i \sqrt{1 + R^2 + 2|R|\cos\left(2\omega\frac{x}{v} + \theta\right)}$$

II-10-3-Maximum and Minimum Amplitude

The amplitude varies between a maximum value U_{max} and a minimum value U_{min}, depending on $\cos\left(2\omega\frac{x}{v} + \theta\right)$:

- Maximum amplitude occurs when $\cos\left(2\omega\frac{x}{v} + \theta\right) = +1$:

$$U_{max} = u_i\,(1 + |R|)$$

- Minimum amplitude occurs when $\cos\left(2\omega\frac{x}{v} + \theta\right) = -1$:

$$U_{min} = u_i\,(1 - |R|)$$

II-10-4-Standing Wave Ratio (SWR)

The standing wave ratio (SWR), denoted by τ\tau$\tau$, is the ratio of the maximum and minimum amplitudes:

$$\tau = \frac{u_{max}}{u_{min}} = \frac{1+|R|}{1-|R|}$$

II-10-5-Physical Interpretation

- When |R|=1 (perfect reflection):
    - $\tau \to \infty$, meaning we have a pure standing wave.

- The amplitude varies between zero and 2Ui at different positions.

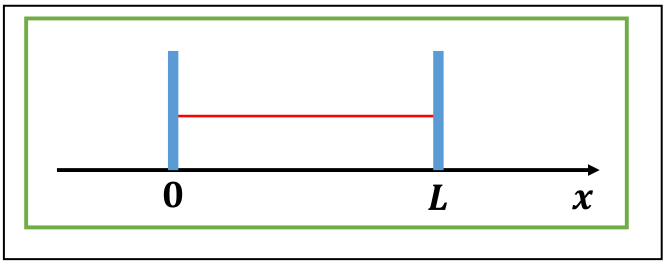
- When |R|=0 (no reflection):
  - τ=1, meaning the wave does not form standing waves at all.
  - The wave is fully transmitted without any reflection.
- For 0<|R|<1:
  - The wave is a combination of a standing and traveling wave.
  - The amplitude oscillates between a finite maximum and minimum.
  - τ indicates the degree of standing wave formation.

II-11- Free and Forced Oscillations of a String of Length L

II-11-1- Free Oscillations of a String Fixed at Both Ends

Consider a string of length LL stretched between two rigid walls at x=0 and x=L. If the string is plucked or disturbed, a sinusoidal wave propagates in both directions, leading to standing waves.

II-11-2- Wave Equation



The general solution for the displacement of the string is:

$u(x,t) = U_i\, e^{j(\omega t - kx)} + U_r\, e^{j(\omega t + kx)}$

where:

- U_i and U_r are the amplitudes of the incident and reflected waves.
- $K = \frac{\omega}{V}$ is the wave number.
- V is the wave velocity.

II-11-3- Boundary Conditions

1. At *x=0*:

The string is fixed, meaning no displacement:

$$u(0,t) = 0\ \forall\, t$$

this gives,

$$Ui + Ur = 0 \Rightarrow\ Ui = -Ur$$

Substituting this into the wave equation:

$$u(x,t) = U_i\ e^{j\omega t}\ (\ e^{-j\omega\left(\frac{x}{V}\right)} - e^{j\omega\left(\frac{x}{V}\right)})$$

Using the identity $e^{-j\theta} - e^{j\theta} = -2j\sin\theta$

$$u(x,t) = 2j\ Ui\ \sin\ (\omega\frac{x}{v}\ )\ e^{j\omega t}$$

Taking the real part:

$$u(x,t) = 2\ Ui\ \sin\ (\omega\frac{x}{v})\ \ sin\,\omega t$$

This is a standing wave with a node at x=0.

2. **At x = L:**

Another fixed point, meaning:

$$u(L,t) = 0\ \forall\ \ t$$

Substituting *x=L* in the equation:

$$U(x,t) = 2\ Ui\ \ sin\ (\omega\frac{x}{v})\ \ sin\,\omega t$$

Here, we recognize the form of a standing wave, which results in the observation of a standing wave phenomenon with a node at x=0.

- Condition at x = L

At x = L, there is a rigid wall. As before, we have:

This condition is satisfied for specific angular frequencies ω_n, called eigenfrequencies:

$$\frac{\omega_n L}{V} = n\pi \Rightarrow\ \ \omega_n = \frac{n\pi\,V}{L}, n = 1,2,3, ...$$

The fundamental frequency corresponds to $n = 1$, and higher values of n correspond to harmonics.

II-11-4- Modes of Vibration

Each mode corresponds to a different wavelength:

- If n = 1: we obtain: $\omega_1 = \ \pi\frac{V}{L}$. Expression $\omega\ and\ V$ as function of frequency

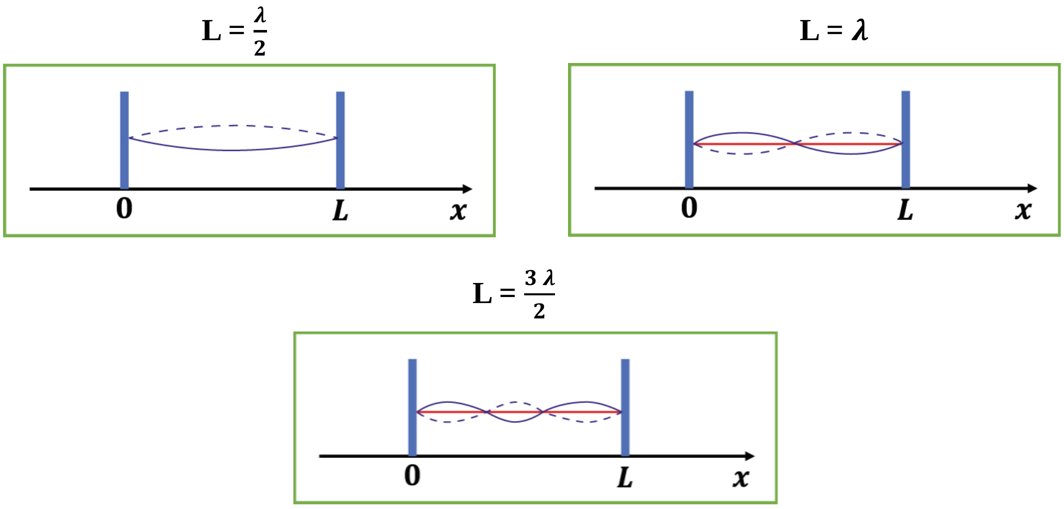
$$\Rightarrow \text{ L} = \frac{\lambda}{2}$$

- If n = 2: we obtain: $\omega_2 = 2\pi\frac{V}{L}$.Expression ω *and* V as function of frequency

$$\Rightarrow \text{ L} = \lambda$$

- If n = 3: we obtain: $\omega_3 = 3\pi\frac{V}{L}$.Expression ω *and* V as function of frequency

$$\Rightarrow \text{ L} = \frac{3\,\lambda}{2}$$

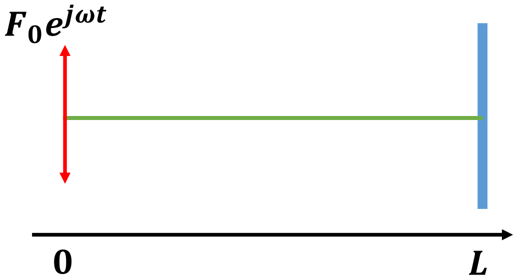


II-12- Forced Oscillations of a String

Now, suppose the string is forced to oscillate at x=0 by an external sinusoidal source:

$$\text{F(t)} = F_0 e^{j\omega t} \ldots\ldots \ (102)$$

II-12-1- Wave Equation



The solution takes the form:

$$\text{u(x,t)} = u_i\ e^{j(\omega t - kx)} + u_r\ e^{j(\omega t + kx))} \ldots\ldots \ (103)$$

where u_i and u_r are unknown coefficients.

II-12-2- Boundary Conditions

1- at ***x=0***:

The wave is excited by an external force:

$$F\ (0,t) = u_i e^{j\omega t} + u_r e^{j\omega t} = u_0 e^{j\omega t}$$

$$u_i + u_r = u_0$$

2- at ***x=L***:

The string is fixed;

$$u(L,t) = u_i\ e^{j\omega\left(t-\frac{L}{V}\right)} + u_r\ e^{j\omega\left(t+\frac{L}{V}\right)} = 0\ \ \forall\, t$$

$$u_i\ e^{-j\omega\frac{L}{V}} + u_r\ e^{j\omega\frac{L}{V}} = 0$$

Solving this system, we get:

$$u(x,t) = u_0\ \frac{\sin\left[\frac{\omega}{V}(L-x)\right]}{\sin\left(\frac{\omega L}{V}\right)} \cos(\omega t)$$

II-12-3- Resonance Condition

When the denominator becomes zero, the amplitude u(x,t) blows up, indicating resonance:

$$\omega\ \frac{L}{V} = n\pi \Longrightarrow\ \omega_n = n\pi\ \frac{V}{L}$$

This matches the natural frequencies of the string, meaning that resonance occurs when the external forcing frequency matches an Eigen-frequency. At resonance, the string vibrates with maximum amplitude.

II-13- Powers of the wave

For a plane sine progressive wave:

$$u(x,t)= A\ \cos(\omega t - kx)$$

The instantaneous power is given by:

$$P(x,t) = \vec{F}(x,t).\dot{\vec{y}}(x,t)$$

Since the force is related to tension T_0:

$$F_y(x,t) = -T_0 \frac{\partial y}{\partial x}$$

And velocity :

$$\dot{y}\,(x,t) = \frac{\partial y}{\partial t}$$

Substituting:

$$P(x,t) = (-T_0)Ak\sin(\omega t - kx).\ (-A\omega)\sin(\omega t - kx)$$

$$P(x,t) = T_0 k\omega A^2 \sin^2(\omega t - kx)$$

Using k=$\frac{\omega}{V}$ and $\frac{T_0}{V}$ = Z_c (characteristic impedance):

$$P(x,t) = z_c\omega^2 A^2\, sin^2(\omega t \pm kx)$$

This equation shows how power varies along the string due to wave propagation.

II-14- Energy Density of Mechanical Waves in a String

For a wave propagating along a stretched string, the total energy consists of kinetic energy and potential energy.

II-14-1- Definition of Energy Density

Energy density (ε) is defined as the energy per unit length of the string:

$$\mathrm{E} = \varepsilon\,\mathrm{dx} \Longrightarrow \varepsilon = \frac{\mathrm{E}}{\mathrm{dx}}\ \left(\frac{\mathrm{j}}{\mathrm{m}}\right)$$

where *E* is the total energy contained in an infinitesimal segment *dx* of the string.

II-14-2- Kinetic Energy Density

The kinetic energy (Ec) of a small element of mass dm moving with velocity $\dot{y}(x,t)$ is:

$$E_c = \frac{1}{2}\mathrm{dm}\ [\dot{y}(x,t)^2]$$

Since the linear mass density of the string is μ, the mass element is:

$$\mathrm{dm} = \mu\,\mathrm{dx}$$

Thus, the kinetic energy in the segment dx is:

$$E_c = \frac{1}{2}\mu\, \mathrm{dx}\, [\dot{y}(x,t)^2]$$

Dividing by *dx* to obtain the kinetic energy density:

$$\varepsilon_c = \frac{E}{dx} = \frac{1}{2}\mu\, [\dot{y}(x,t)^2]$$

II-14-3- Potential Energy Density

The potential energy (*Ep*) comes from the stretching of the string due to the transverse displacement *y(x,t)*. The length of the infinitesimal string segment changes due to the wave.

 In equilibrium, the segment has a length dx.

 When displaced, its new length dl is:

$$l = \frac{dx}{\cos\theta}$$

Since : $\cos\theta = \frac{1}{(1+(\frac{\partial y}{\partial x})^2)^{\frac{1}{2}}}$, we use the first-order approximation for small slopes

$$(1-s)^n = (1-ns) : d\,l \approx \ dx\,(1+\frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2)$$

The increase in length due to the wave is:

$$\mathrm{dl} - \mathrm{dx} \ \approx \ \mathrm{dx}\ \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2$$

The tension force T_0 does work due to this elongation, leading to potential energy:

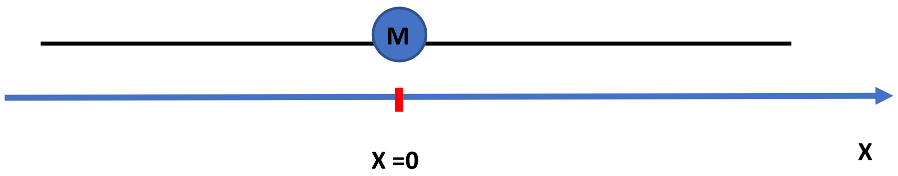
$$E_p = \ T_0\,(dl - dx) = \frac{T_0}{2}(\frac{\partial y}{\partial x})^2 dx$$

Dividing by *dx* gives the potential energy density:

$$\varepsilon_p = \ \frac{T_0}{2}(\frac{\partial y}{\partial x})^2$$

Exercises and solved problems

Exercise 1: An infinite string with a linear mass density μ is subjected to a constant tension T_0. A point mass M is attached to the string at x = 0. A wave with angular frequency ω arrives from $-\infty$ and propagates in the positive x direction. The weight of the string is neglected to study the small displacements y(x,t) of the string.



1- Write the expressions for y_1 (x,t) for x<0 and y_2 (x,t) for x>0 in terms of the reflection coefficient R and the transmission coefficient T for the displacement at x=0.

2- Using the continuity of displacements at x = 0 and applying the fundamental relation of dynamics to the mass M, determine the reflection coefficient R at x = 0

3- What are the limiting values of R when M $\longrightarrow$ 0 et M $\longrightarrow \infty$?

Solution:

1) y_1(x,t) = $y_i\ e^{j(\omega t-kx)} + y_r\ e^{j(\omega t+kx)}$ and y_2(x,t) = $y_t\ e^{j(\omega t-kx)}$ with $k = \frac{\omega}{V}$, with V the propagation velocity of the waves in the string.

Starting from the definitions of the reflection and transmission coefficients, we have:

$$R = \frac{y_r}{y_i}\ and\ T = \frac{y_t}{y_i}$$

This allows us to write:

$$y_1(x,t) = y_i\left[e^{j(\omega t-kx)} + R\, e^{j(\omega t+kx)}\right] \text{ and } y_2(x,t) = y_i T\ e^{j(\omega t-kx)}$$

2) **Continuity of displacement at** *x=0:*

$$y_1(0,t) = y_2(0,t) \Rightarrow 1 + R = T \ldots\ldots..(1)$$

Applying the fundamental relation of dynamics:

From the fundamental dynamic relation (FDR), we have:

$$Z(0) = Z_c + jM\omega \ldots\ldots\ldots\ldots\ldots..(2)$$

Thus, we find:

$$R = \frac{Z_c - Z(0)}{Z_C + Z(0)}$$

3)

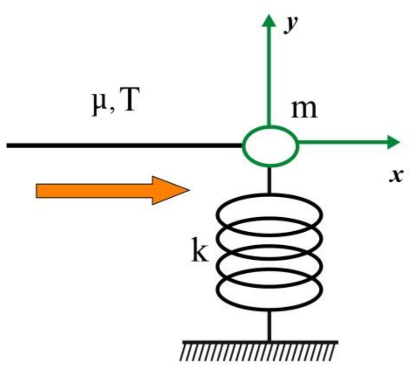
- When M→0, we get R→0, meaning there is no reflection, corresponding to the case of an infinite string.
- When M→∞, we get R→−1, meaning the end of the string behaves like a fixed point at x=0, leading to total reflection with an amplitude inversion.

Exercise 2: A homogeneous string with linear mass density μ is stretched with a tension T much greater than its weight, along the axis x′Ox from −∞ to x=0. At x=0, it is terminated by a spring with stiffness K and a negligible mass m. A transverse wave is sent from −∞ and propagates along the string. We write: y(x,t) = A $e^{j(\omega t - kx)}$.

1- By applying the fundamental relation of dynamics, determine the terminal impedance Z_T at x=0 and deduce the value of the reflection coefficient at this point, specifying its magnitude and phase.

2- Study the behaviour of the system in the case where $\omega = \sqrt{\frac{k}{m}}$.

3- What will be observed when m is infinite k infinite?



Solution:

1) At x=0, there are two forces:

- The force exerted by the string (left part).
- The force exerted by the spring.

Applying the fundamental principle of dynamics (projection along the y axis) gives:

$$- T\,\frac{\partial y}{\partial x} - ky = m\ddot{u}$$

We could express $y\ and\ \ddot{y}$ in terms of the transverse velocity:

$$\ddot{y} = j\omega\dot{y}\ and\ y = \frac{k}{j\omega}\dot{y}$$

By substituting into the previous equation and then simplifying with respect to $\dot{y}$, we obtain:

$$-T\,\frac{\frac{\partial y}{\partial x}}{\dot{y}} = j\left(m\omega - \frac{k}{\omega}\right) = Z_T = Z\,(0)$$

Thus, the reflection coefficient is:

$$\mathrm{R} = \frac{\mathrm{Z_c} - \mathrm{Z_T}}{\mathrm{Z_c} + \mathrm{Z_T}} = \frac{\mathrm{Z_c} - j\left(m\omega - \frac{k}{\omega}\right)}{\mathrm{Z_c} + j\left(m\omega - \frac{k}{\omega}\right)}$$

We deduce that: $|R| = 1$ and $Arg(R) = -2\,artg\,\frac{\left(m\omega - \frac{k}{\omega}\right)}{Z_c}$

2) if $\omega = \sqrt{\frac{k}{m}}$, then R = +1, it behaves as if there were a free end at x = 0: the wave does not see obstacle, and the masse spring system acts as a freely oscillating autonomous system. This results in phenomenon of stationary waves on the string, with an atinodes at x =0.

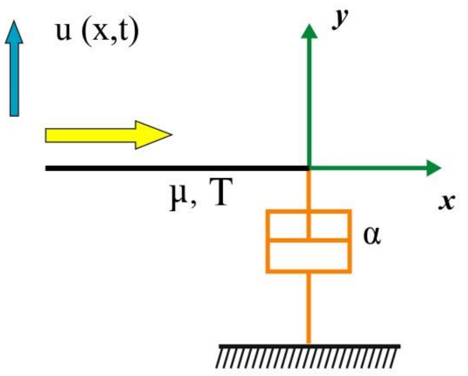
3) I m or k is infinit, it behaves as if there were a rigid wall at x=0: there is a total reflection with a phase shift (R = -1) creating a node at x =0.

Exercise 3: A homogeneous string with linear mass density μ is stretched with a tension T much greater than its weight, along the axis x′Ox from −∞ to x=0. At x=0, it is terminated by a damper with a viscous damping coefficient α. A transverse wave is sent from −∞ and propagates along the string. We write: y(x,t) = A $e^{j(\omega t - kx)}$..

1- By applying the fundamental relation of dynamics, determine the terminal impedance Z_T at x=0 and deduce the value of the reflection coefficient at this point, specifying its magnitude and phase.

2- Give the expression for the resulting displacement y(x,t) of a point x on the string in the case where μT>α and write it in the form y(x,t) = $Y_0 e^{j(\omega t-kx)}$. Express Y_0 in terms of yi , and Y(x) in terms of R, Yi, k, and x.

3- Discuss these cases where $\alpha = 0$, α infinite and $\alpha = \sqrt{\mu T}$.



Solution:

1) At x=0, there are two forces:

- The force exerted by the string (left part).
- The force exerted by the spring.

Applying the fundamental principle of dynamics (projection along the y axis) gives:

$$-T\,\frac{\partial y}{\partial x} - \alpha\dot{y} = 0$$

By dividing by $\dot{y}$, we obtain:

$$\left[-T\,\frac{\frac{\partial y}{\partial x}}{\dot{y}} - \alpha\right]_{x=0} = 0 \;\Rightarrow\; -T\,\frac{\frac{\partial y}{\partial x}}{\dot{y}} = \alpha$$

The first term of the second equality represents the impedance of the string at the point x=0, that is, Z(0) = Z_T.

Thus, we obtain: $Z_T = \alpha$

Thus, the reflection coefficient is:

$$R = \frac{Z_c - \alpha}{Z_c + \alpha}$$

- If $Z_c > \alpha$, we have:

$|R| = \frac{Z_c-\alpha}{Z_c+\alpha}$, then: $\arg(R) = 0$.

- If $Z_c < \alpha$, we have:

$|R| = \frac{\alpha - Z_c}{\alpha + Z}$, then: $\arg(R) = \pi$.

2) The general expression of $y(x, t)$ is written as:

$$y\,(x,t) = y_i\left[e^{j(\omega t-kx)} + R\,e^{j(\omega t+kx)}\right]$$

Factoring out $e^{j(\omega t-kx)}$, we obtain:

$$y\,(x,t) = y_i\left[1 + R\,e^{j2kx}\right]e^{j(\omega t-kx)}$$

We deduce :

$Y_0 = Y_i$ and $U(x) = 1 + R\,e^{j2kx}$

3)

- If α= $\sqrt{\mu T}$, then R = 0, there is no reflection (R=0): everything behaves as if the string were infinite.
- If α is infinite, R = -1: total reflection with phase change. This behaves as if there were a rigid wall at x =0 (a barrier). A phenomenon of stationary waves occurs with a node at x =0.

- If α is 0, R = +1: total reflection without phase change. This behaves as if there were a free end ar x =0. A phenomenon of stationary waves occurs with a antinode at x =0.

Exercise 4: Let there be a homogeneous string of length L, with linear mass density μ\muμ, subjected to a tension T_0. A plane harmonic wave (monochromatic with angular frequency ω) propagates along this string. The solution to its equation of motion (the propagation equation) gives the transverse displacements

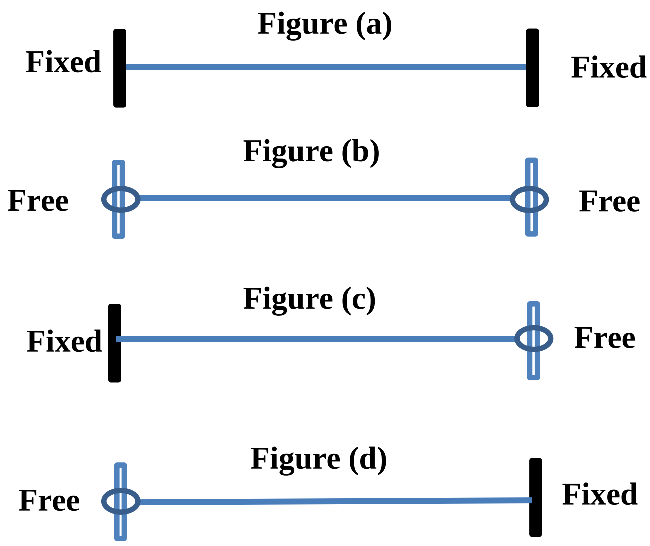
$$y(x,t) = A\, e^{i(\omega t - kx)} + B\, e^{i(\omega t + kx)}$$

A and B are two constants, and k represents the magnitude of the wave vector.

1- Write the boundary conditions at x=0 and x=L for cases (a) and (b). Deduce from the obtained relations the equations for the natural frequencies (eigenfrequencies). Establish the relationship between the wavelength λ and the length L of the string.

2- Draw the appearance of the string (a) and (b) for the fundamental mode as well as the first harmonic.

3- Write the boundary conditions with respect to the impedances Z(0) and Z(L) for figure (c), and with respect to the coefficients R(0) and R(L) for the string in figure (d). Draw the appearance of the string (c) for the second harmonic. Find the frequency of the third harmonic of the string (d). Conclusion and remarks?

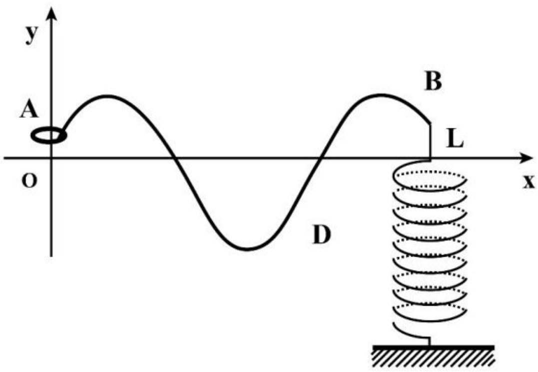


Exercise 5: Consider a string of length L and linear mass μ stretched horizontally with constant tension T. One of its ends (A), at abscissa x=0, is free. One end (A), abscissa x=0, is free. The other end (B), abscissa x=L, is connected to a spring of stiffness C, whose other end is connected to the ground. V is the velocity of propagation of transverse waves along the string. The displacement at any point x on the rope is expressed as:

$$y(x,t) = Ae^{j(\omega t - kx)} + B\, e^{j(\omega t + kx)} \text{ with } k = \frac{\omega}{V}$$

1- By writing the condition at x = 0, show that standing waves are established, described by a transverse displacement of the form y(x,t)=f(x)g(t) where f(x) is a function to be explained.

2- By writing the boundary condition at x=L, deduce the equation for the natural frequencies.



3- Draw the shape of the string when it vibrates at its lowest frequency:

a- In the case where D = 0.

b- In the case where D is infinite.

Exercise 6: A string of length L and linear mass density μ is stretched with a tension T. It is attached to a fixed support at the end x=L. The end at x=0 is subjected to a sinusoidal force of amplitude F_0 and angular frequency ω. Let y(x,t) represent the displacement of a point at position x on the string at time t.

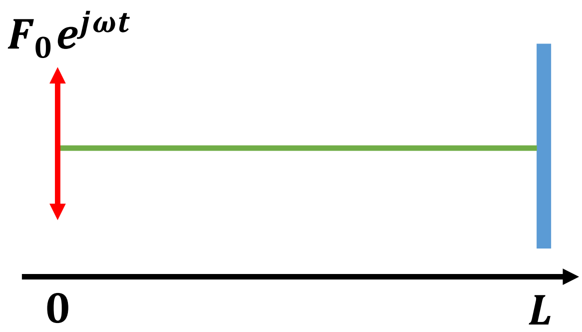
1- a- Provide the boundary conditions at x=0 and x=L.

b- Show that y(x,t) can be expressed as : y(x,t) $= \psi(x)\ \phi(t)$.

c- Give the expression ψ(x) as a function of F_0, the wavenumber k, L and T

2- Determine the positions of the points of maximum amplitude (antinodes) as a function of the wavelength λ and the length L of the string. What is the distance between two successive nodes.

3- What must be the angular frequency of the force f(t) to observe the resonance phenomenon.



Chapter III: Acoustic waves in fluids

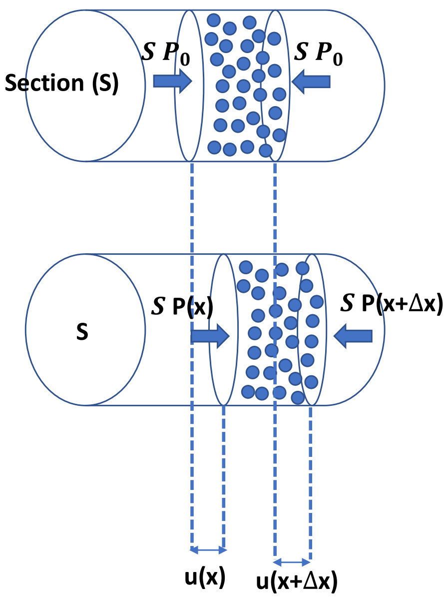
III- Introduction

Acoustic waves are low-amplitude longitudinal waves (typically a few millimeters at most) that propagate primarily in fluids or solids. Depending on their frequency, they may or may not be audible to the human ear; for instance, ultrasound is inaudible to humans but can be perceived by certain animals

This disturbance is accompanied by variations in pressure and density that propagate from one point to another.

III-1- The propagation equation of waves

Consider a semi-infinite tube with a cross-sectional area S, aligned along the x′Ox axis, extending from x=0 to infinity, and filled with a viscous fluid of equilibrium density ρ\rhoρ and equilibrium pressure P_0 . We analyze a small fluid slice between x and x+Δx. When the fluid is at rest, this slice remains stationary because the forces acting on it from both sides cancel each other out.



However, if a disturbance is introduced at the tube's entrance, it propagates along the tube. The particles initially at position x experience a displacement (x), while those initially at x+Δx move by $(x$+$\Delta x)$. The **initial volume** of the slice is: v_0=$S\Delta x$

1. New Volume Expression

Due to the displacement of the fluid particles, the new volume of the slice becomes:

$$v = \{[\ x+\Delta x + (u(x) + \Delta x)] - [x + u(x)]\} = S\,[\Delta x + u\,(x + \Delta x) - u(x)]$$

For small displacements, we approximate: $v = \ S\,\Delta x + S\,\Delta x\,\frac{\partial u}{\partial x}$

Thus we deduce:

$$\frac{\partial u}{\partial x} = \frac{v - v_0}{v_0}$$

2. Fundamental Equation of Dynamics

As the disturbance passes, the pressure at position x becomes P(x), and at x+ Δx it becomes P(x+ Δx). The corresponding pressure variations are:

$$p(x) = P(x) - P_0 \;\ldots\ldots\; (5) \text{ and } p\,(x+\Delta x) = P(x+\Delta x) - P_0$$

P (x) referred to as the suppression at the abscissa point x.

Applying Newton's second law to the slice of fluid between x and x+Δx:

$$S\big(\mathrm{p(x)} - \mathrm{p(x+\Delta x)}\big) = \mathrm{dm}\,\frac{\partial \dot{\mathrm{u}}}{\partial \mathrm{t}}$$

Using the Taylor expansion:

$$S\left(-\frac{\partial P}{\partial x}\right) dx = \rho\, S\, dx\, \frac{\partial \dot{u}}{\partial t}$$

Since $\rho = \frac{dm}{S\,dx}$ we obtain:

$$\frac{\partial p}{\partial x} + \rho\,\frac{\partial \dot{u}}{\partial t} = 0$$

This is the fundamental equation of dynamics.

Fundamental Equation of Acoustics

The variation in volume of the slice causes a corresponding variation in pressure, given by:

$$p(x,t) = -\frac{1}{\chi}\,\frac{\partial u}{\partial x}$$

Where χ is the adiabatic compressibility coefficient of the fluid.

3. Wave Equations

Substituting *P(x,t)* into the previous equation, we derive the wave equation for particle displacement:

$$\frac{\partial^2 u}{\partial x^2} - \rho\chi\frac{\partial^2 u}{\partial t^2} = 0$$

Similarly, the wave equation for pressure propagation is:

$$\frac{\partial^2 p}{\partial x^2} - \rho\chi \frac{\partial^2 p}{\partial t^2} = 0$$

4. Phase Relationship Between Pressure and Displacement

Since equation (2) shows a derivative relationship between *p(x,t)* and *u(x,t)*, in the case of a sinusoidal wave, pressure and displacement are always in quadrature. This means:

- When the displacement is at its maximum or minimum (u(x,t)= ±1), the pressure is zero (*p(x,t)=0*).
- When the displacement is zero, the pressure is at its maximum or minimum.

This phase shift is a key characteristic of wave behavior in fluids.

III-2- Continuity Equation (Mass Conservation Equation)

In fluid mechanics and wave propagation, the continuity equation expresses the conservation of mass. It ensures that as a wave propagates through a medium, mass is neither created nor lost. We consider a small fluid slice between *x* and *x+dx* in a tube of cross-sectional area *SSS*, where the fluid has a density *ρ* and a velocity field *u(x,t)*.

III-2-1- Time Rate of Change of Mass

The total mass of the fluid element is:

$$m=\rho S dx$$

The rate of change of mass in the control volume is given by:

$$\frac{\partial m}{\partial t} = S\, dx\, \frac{\partial \rho}{\partial x}$$

III-2-2- Mass Flux Through the Boundaries

The mass leaving the control volume through *x+dx* is different from the mass entering at *x*. This difference in mass flux accounts for changes in mass within the control volume.

The difference in mass flux between the two positions is:

$$\mathrm{dm} = \mathrm{m(x)} - \mathrm{m(x+dx)}$$

Since mass flux is given by $Ps\dot{u}$, we write:

$$dm = \rho S\, dt\, [\dot{u}(x) - \dot{u}(x + dx)]$$

Using a first-order Taylor expansion:

$$dm = \rho S\, dt \left(-\frac{\partial \dot{u}}{\partial x}\right) dx$$

III-2-3- Mass Conservation Principle

Since mass is conserved, the change in mass within the element must equal the net mass flux through the boundaries:

$$\frac{\partial \mathrm{m}}{\partial \mathrm{t}} + \frac{\partial \mathrm{m}}{\partial \mathrm{t}} = 0$$

Substituting Equations (13) and (17):

$$\frac{\partial \mathrm{m}}{\partial \mathrm{t}} = \mathrm{Sdx}\, \frac{\partial \rho}{\partial \mathrm{t}} = -\, \rho \mathrm{S\, dx}\, \frac{\partial \dot{\mathrm{u}}}{\partial \mathrm{x}}$$

Dividing by SdxS (assuming nonzero values):

$$\frac{\partial \rho}{\partial \mathrm{t}} + \rho\, \frac{\partial \dot{\mathrm{u}}}{\partial \mathrm{x}} = 0$$

This is the continuity equation (or mass conservation equation), which ensures that any compression of the fluid (change in ρ\rhoρ) is accompanied by a corresponding velocity divergence $\frac{\partial \dot{u}}{\partial x}$.

III-3- Wave Propagation Velocity

The velocity of wave propagation in a fluid is given by:

$$\mathrm{V} = \frac{1}{\sqrt{\rho \chi}}$$

where:

- ρ is the density of the fluid,
- χ is the adiabatic compressibility coefficient.

Using this, the wave equation for pressure propagation can be rewritten as:

$$\frac{\partial^2 p}{\partial x^2} - \frac{1}{V^2}\frac{\partial^2 p}{\partial t^2} = 0$$

Thus, the velocity of sound propagation in a fluid is given by:

$$V = \sqrt{\frac{\gamma P_0}{\rho}}$$

where:

- γ is the ratio of specific heats (at constant pressure and constant volume).
- P_0 is the equilibrium pressure of the fluid.
- ρ is the density of the fluid.

III-3-1- Velocity of Sound in Air

For air at $T = 0°C$:

- $\gamma = 1.4$,
- $\rho = 1.225 \frac{kg}{m^3}$,
- $P_0 = 1.013hPa$.

Substituting these values:

$$V = 340.25 \frac{m}{s}$$

III-3-2- Velocity of Sound in an Ideal Gas

For an ideal gas, the equation of state is:

$$P_0 v_0 = RT$$

Where :

- γ: Ratio of specific heats at constant pressure and constant volume
- R : The ideal gas constant,
- T: Temperature in Kelvin,
- M : The molar mass of the gas

From this, we express the velocity of sound as:

$$P_0 = \frac{\mathrm{RT}}{v_0} \Rightarrow V = \sqrt{\frac{\gamma P_0}{\rho}} = \sqrt{\frac{\gamma RT}{\rho v_0}} = \sqrt{\frac{\gamma RT}{M}}$$

where *M* is the molar mass of the gas.

III-4- Solution de l'équation de propagation

The general solution of the wave equation is given by:

$$P(x,t) = F\left(t - \frac{x}{V}\right) + G\left(t + \frac{x}{V}\right)$$

Where:

- $F\left(t - \frac{x}{V}\right)$ represents a wave propagating in the positive x-direction (right-moving wave),
- $G\left(t + \frac{x}{V}\right)$ represents a wave propagating in the negative x-direction (left-moving wave).

III-5- Harmonic plane progressive waves

In the case of a harmonic plane wave, the pressure solution takes the form:

$$P(x,t) = P_0 e^{j(\omega t - kx)}$$

where:

- P_0 is the amplitude of the pressure wave,
- Ω is the angular frequency,
- $K = \frac{\omega}{V}$ is the wave number,
- j is the imaginary unit ($j^2 = -1$).

The displacement velocity of the particles in the medium is obtained using the relation:

$$\dot{u}(x,t) = \frac{\partial u_x}{\partial t} = j\omega u_x = \frac{1}{\rho V} P_0 e^{j(\omega t - kx)}$$

which shows that the particle velocity is proportional to the pressure wave and is shifted in phase.

We obtain u(x,t) by integrating with respect to x based on relation (10), and then differentiating with respect to time:

$$\dot{u}(x,t) = \frac{d}{dt}\left[-\chi \int p(x,t)\ dx\right]$$

III-6-Impedance at a point

The impedance at a point x of an acoustic wave is the ratio between the overpressure p(x,t) and the particle velocity at that point:

$$Z(x) = \frac{P(x,t)}{\dot{u}(x,t)}$$

Acoustic impedance is expressed in Rayleighs (Ra or Rayl) in honor of the English physicist J. W. S. Rayleigh (1842-1919):

1 Ra=1 Pa/m/s.

In the case of a progressive or regressive wave, the impedance is given by:

$$Z(x)= \rho V = \sqrt{\frac{\rho}{\chi}}\ \forall\ x$$

This constant value is called the characteristic impedance, just as in the case of vibrating strings.

III-7- Propagation of energy

III-7-1- Kinetic Energy Density

Consider a small volume element v_0 (a particle) moving under the action of an acoustic wave with displacement u(x,t).

The velocity of the particles is:

$$\dot{u}(x,t) = \frac{\partial u_x}{\partial t} = j\omega u_x = \frac{1}{\rho V} P_0 e^{j(\omega t - kx)}$$

The kinetic energy of the particles is given by:

$$E_c = \frac{1}{2}\ dm\ \dot{u}_x 2\ = \frac{1}{2}\rho v_0\ \dot{u}_x^2$$

Thus, the kinetic energy density is: $\varepsilon_c = \frac{E_c}{v_0} = \frac{1}{2}\rho \dot{u}_x^2$

III-7-2- Potential energy density

The work required by the acoustic wave to compress a small volume element by an amount dV:

$$dW = P\,dV$$

The corresponding change in potential energy is:

$$dE_p = -dW = -P\,dv$$

From the equation of state:

$$dp = p = -\frac{1}{\chi}\frac{v - v_0}{v_0} \Longrightarrow dp = -\frac{1}{\chi}\frac{dv}{v_0} = -\rho V^2 \frac{dv}{v_0}$$

Integrating from 0 to P, we obtain the potential energy:

$$E_p = \frac{v_0}{\rho V^2}\int_0^P P dp = \frac{1}{2\rho V^2} P^2 v_0$$

Thus, the potential energy density is: $\varepsilon_p = \frac{1}{2\rho V^2} p^2$

III-7-3- Total Energy Density

The total energy density of the wave is given by the sum of the kinetic and potential energy densities:

$$\varepsilon = \varepsilon_c + \varepsilon_p = \frac{1}{2}\rho \dot{u}_x 2 + \frac{1}{2\rho V^2} p^2$$

For a sinusoidal progressive wave, we have:

$$\mathrm{P(x,t)} = P_0 e^{j(\omega t - kx)}$$

$$\dot{u}(\mathrm{x,t}) = \frac{\partial \mathrm{u_x}}{\partial \mathrm{t}} = j\omega u_x = \frac{1}{\rho V} P_0 e^{j(\omega t - kx)}$$

Thus, the energy density becomes:

$$\varepsilon = \varepsilon_c + \varepsilon_p = \frac{p_0^2}{\rho V^2}\cos^2(\omega t - kx)$$

Taking the average value: $\langle \varepsilon \rangle = \frac{p_0^2}{2\rho V^2}$

III-8- Power flux - Intensity

$$dE = \varepsilon\, S\, V\, dt$$

$$P = \frac{dE}{dt} = \varepsilon\, S\, V$$

$$I = \frac{P}{S} = \varepsilon\, V = \frac{P_0^2}{\rho V}\cos^2(\omega t - kx)$$

Taking the average value of the acoustic intensity:

$$\langle I\rangle = \frac{\langle P\rangle}{S} = \langle\varepsilon\rangle\, V = \frac{P_0^2}{2\rho V} = \frac{P_0^2}{2Z_c}$$

The sound level in decibels (dB) is defined as:

$$N_{dB} = 10\log 10\left[\frac{I}{I_0}\right]$$

Where:

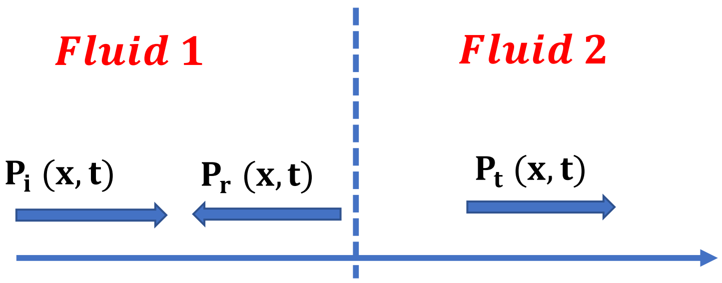
$$I_0 = \frac{10^{-12}W}{m^2}$$

is the reference intensity, which corresponds to the threshold of hearing, the minimum intensity that the human ear can hear.

III-9- Reflection and transmission

A progressive sinusoidal plane acoustic wave of pressure propagates along the $x'Ox$ axis in the positive direction.

- The amplitude is denoted by P_i.
- The angular frequency is denoted by ω.



The medium is divided into two regions:

- For x<0 : The wave propagates in a fluid (1) characterized by (ρ_1, V_1, Z_{C1}).
- For x>0: The wave propagates in another fluid (2) characterized by (ρ_2 ,V_2 ,Z_{C2}).

III-9-1- Reflection and Transmission Coefficients

- In terms of pressure: $R_P = \frac{p_r}{p_i}$ et $T_P = \frac{p_t}{p_i}$
- In terms of amplitude: $R_u = \frac{U_r}{U_i}$ et $T_u = \frac{U_t}{U_i}$

- In terms of intensity: $R_I = \frac{I_r}{I_i}$ et $T_I = \frac{I_t}{I_i}$

III-9-2- Wave Equations in Both Media

Medium 1 $x < 0$:

$$\begin{cases} P(x,t) = A\, e^{j(\omega t - k_1 x)} + B\, e^{j(\omega t + k_1 x)} \\ \dot{u}_1 = \dfrac{1}{Z_1}\left[A\, e^{j(\omega t - k_1 x)} - B\, e^{j(\omega t + k_1 x)}\right] \end{cases}$$

Medium 2 $x \geq 0$:

$$\begin{cases} P(x,t) = C\, e^{j(\omega t + k_2 x)} \\ \dot{u}_1 = \dfrac{1}{Z_2}\left[C\, e^{j(\omega t + k_2 x)}\right] \end{cases}$$

- **Continuity Conditions at x=0**

1- Continuity of pressure:

$$A + B = C\ldots\ldots\ldots\ldots\ldots(1)$$

2- Continuity of particle velocity:

$$\frac{1}{Z_1}[A - B] = \frac{1}{Z_2}C\ldots\ldots(2)$$

- **Reflection and Transmission Coefficients**

Solving for R_p and T_p :

For Pressure:

$$R_p = \frac{B}{A} = \frac{Z_2 - Z_1}{Z_2 + Z_1}, T_p = \frac{C}{A} = \frac{2\, Z_2}{Z_2 + Z_1}$$

For the intensity:

$$\alpha_R = \frac{I_R}{I_I} = \frac{(Z_2 - Z_1)^2}{(Z_2 + Z_1)^2}, \alpha_T = \frac{I_T}{I_I} = \frac{4\, Z_1 Z_2}{(Z_2 + Z_1)^2}$$

III-10-Reflection and transmission in tubes

We consider a progressive plane acoustic wave propagating in a tube with a cross-sectional area S.

- Acoustic Flow and Impedance

The acoustic flow is defined as: $d = S\,\dot{u}$, where $\dot{u}$ represents the velocity of the fluid particles.

The impedance at a point x in the tube is given by:

$$Z_{(x)} = \frac{P(x,t)}{S\,\dot{u}(x,t)} = \frac{P(x,t)}{d\,(x,t)}$$

If a tube with cross-sectional area S is traversed by a sinusoidal acoustic pressure wave with amplitude P_0 and angular frequency ω, propagating in the positive x direction, it can be shown that:

$$Z_{(x)} = \frac{\rho v}{s} = Zc$$

Where Zc is the characteristic impedance of the tube.

- **Reflection and Transmission at an Interface**

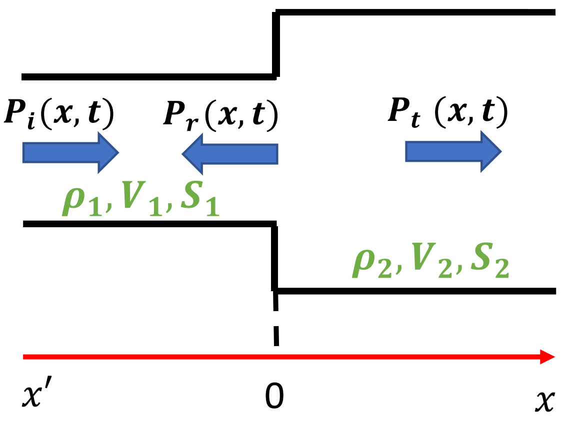
If an incident acoustic wave propagates in a tube with cross-sectional area S_1 (filled with a fluid of density ρ_1), connected to another tube with cross-sectional area S_2 (filled with a fluid of density ρ_2), the reflection and transmission coefficients at the interface x=0 can be determined using the following procedure:

- **Expression for the Pressure Waves**

The pressure of the incident wave is expressed as: P_i (x, t) = A $e^{j(\omega t - k_1 x)}$

For x<0 (before the interface), the total pressure is: P_1 (x, t) = A $e^{j(\omega t - k_1 x)}$ + A $e^{j(\omega t + k_1 x)}$

For x>0 (after the interface), the pressure is: P_2 (x, t) = A $e^{j(\omega t - k_2 x)}$



Boundary Conditions at x=0

Continuity of pressures at x=0 is expressed as:

$$P_1(0, t) = P_2(0, t)$$

Continuity of acoustic flow at x=0 is expressed as:

$$d_1(0, t) = d_2(0, t)$$

Using these conditions, we obtain the reflection and transmission coefficients:

- **Reflection coefficient**:

$$R_p = \frac{Z_{c2} - Z_{c1}}{Z_{c2} + Z_{c1}}$$

- **Transmission coefficient**:

$$T_p = \frac{2Z_{c2}}{Z_{c2} + Z_{c1}}$$

where the characteristic impedances are defined as:

$$Z_{c1} = \frac{\rho_1 v_1}{S_1} \text{ et } Z_{c2} = \frac{\rho_2 v_2}{S_2}$$

III-11-The Doppler effect

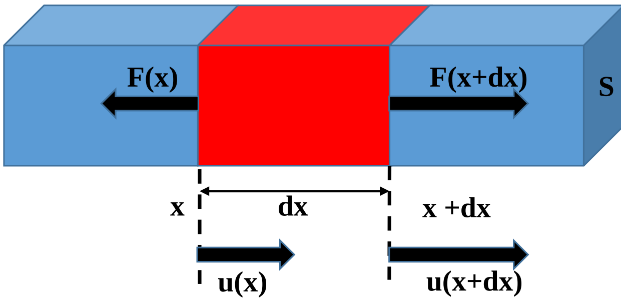
The Doppler effect is a phenomenon related to the motion of the wave source or the observer (wave detector). When an observer is sitting by the road and a car passes by at high speed while honking, the frequency of the sound perceived is different from what the observer would perceive if the car were stationary. This same effect is also observed when the car is stationary and the observer moves at high speed. This phenomenon is also observed when both the source and the observer are moving relative to each other, in general.

This effect was described by Christian Doppler (in 1842) and Hippolyte Fizeau (in 1848).

Chapter IV: Elastic waves in solids

IV-1- Introduction

Consider a low-amplitude acoustic wave propagating along the x direction in a continuous and isotropic deformable solid with cross-sectional area S and mass density ρ.



During its propagation, the acoustic wave is accompanied by infinitesimal variations in:

- Strain: ε(x,t),
- Stress: τ(x,t).

IV-2-Hooke's law

The relationship between stress and strain in an elastic solid is given by Hooke’s Law:

$$\varepsilon = \frac{1}{E}\tau \text{ and } \tau = \frac{F}{S}$$

Since strain is defined as the spatial derivative of displacement u(x,t), we have:

$$\varepsilon = \frac{\partial u}{\partial x}$$

where:

- E is Young’s modulus, a material-specific constant that characterizes the stiffness of the solid.

IV-3-Characteristics of some solids

The following table provides density and Young's modulus for various solid materials:

Solids	Density (kg.m^{-3})	Young'modulus (Gpa)
Steel	**7850**	**210**
Aluminum	**2700**	**62**
Concrete	**2200-2500**	**20-40**
Alumina	**3950**	**3.1**

IV-4-Derive the d'Alembert equation

- **Forces Acting on a Small Element**

Consider a small element of length dx within an elastic solid. The forces at the two ends are given by:

$$F\,(x+dx) = S\,\tau\,(x+dx) = S\,E\,\varepsilon\,(x+dx) = S\,E\,\frac{\partial u}{\partial x}\Big|_{x+dx}$$

$$F\,(x) = -\,S\,\tau\,(x) = -\,S\,E\,\varepsilon\,(x) = -\,S\,E\,\frac{\partial u}{\partial x}\Big|_{x}$$

- **Applying Newton's Second Law**

The fundamental relation of dynamics states that the net force on the element equals its mass times acceleration:

$$S\,E\,\frac{\partial u}{\partial x}\Big|_{x+dx} - S\,E\,\frac{\partial u}{\partial x}\Big|_{x} = m\,\frac{\partial^2 x}{\partial t^2}$$

Since the mass of the element is:

$$m = \rho\,v = \rho\,S\,dx$$

Substituting m in the equation:

$$S\,E\,\frac{\partial u}{\partial x}\Big|_{x+dx} - S\,E\,\frac{\partial u}{\partial x}\Big|_{x} = \rho\,S\,dx\,\frac{\partial^2 x}{\partial t^2}$$

Dividing by Sdx, we get:

$$E \frac{\partial^2 x}{\partial x^2} = \rho \frac{\partial^2 x}{\partial t^2}$$

Rearranging:

$$\frac{\partial^2 x}{\partial x^2} - \frac{1}{V^2}\frac{\partial^2 x}{\partial t^2} = 0$$

where the wave propagation velocity is:

$$V = \sqrt{\frac{E}{\rho}}$$

IV-5-Solution of the Wave Equation

The general solution of the one-dimensional wave equation is:

$$u(x,t) = F\,(t - \frac{x}{V}) + G\,(t + \frac{x}{V})$$

where:

- $F\,(t - \frac{x}{V})$ represents a wave propagating in the positive x direction.
- $G\,(t + \frac{x}{V})$ represents a wave propagating in the negative x direction.

IV-6- Impedance at a Point

The impedance at a point is defined as the ratio of the complex amplitude of the force to the complex amplitude of the particle velocity**:**

$$Z\,(x) = \frac{F\,(x,t)}{\dot{u}(x,t)}$$

For a progressive wave, the force is:

$$F\,(x,t) = -\,S\,E\,\frac{\partial u}{\partial x}$$

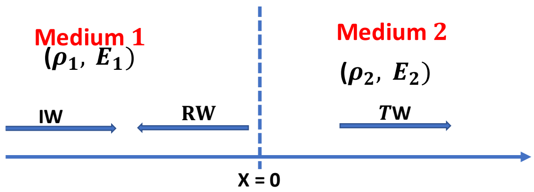
For a progressive sinusoidal longitudinal plane wave, the impedance is: $Z(x)= S\,\sqrt{E\,\rho}$

The characteristic impedance is then: $Z_c = \sqrt{E\,\rho}$.

IV-7-Reflection and Transmission at an Interface

Consider two media:

- Medium 1 ($x < 0$) with characteristics (ρ_1, E_1).
- Medium 2 1 ($x \geq 0$) with characteristics (ρ_2, E_2).



Medium 1 x < 0:

$$u_1(x,t) = A\, e^{j(\omega t - k_1 x)} + B\, e^{j(\omega t + k_1 x)} \textit{ and } F_1(x,t) = j\omega S\sqrt{\rho_1 E_1}\left[A\, e^{j(\omega t - k_1 x)} - B\, e^{j(\omega t + k_1 x)}\right]$$

Medium 2 x ≥ 0:

$$u_2(x,t) = C\, e^{j(\omega t + k_2 x)} \text{ and } F_2(x,t) = j\omega S\sqrt{\rho_2 E_2}\left[C\, e^{j(\omega t +\, k_1 x)}\right]$$

- **Boundary Conditions at x=0:**

Continuity of particle velocity:

$$A + B = C\text{……………}(1)$$

- **Force balance (Newton's second law):**

$$Z_1[A - B] = Z_2 C\text{…….}(2)$$

- **Reflection and Transmission Coefficients**

Solving for the reflection coefficient:

$$R = \frac{B}{A} = \frac{Z_1 - Z_2}{Z_1 + Z_2}$$

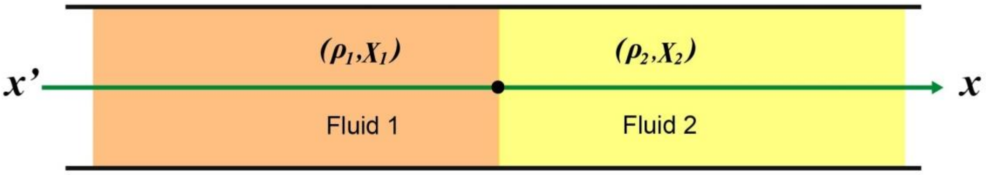
And the transmission coefficient:

$$T = \frac{C}{A} = \frac{2\, Z_1}{Z_2 + Z_1}$$

Exercises and solved problems

Exercise 1

An infinitely long cylindrical tube with a constant cross-section S contains two fluids (1) and (2) with densities ρ_1 and ρ_2, respectively. The wave propagation speeds in these fluids are V_1 and V_1. The coordinate origin is set at x = 0, so that fluid (1) occupies the region x<0 and fluid (2) occupies x>0.



A plane progressive acoustic pressure wave propagates in fluid (1) as:

$$p_i(x,t) = Ae^{j(\omega t - kx)}$$

where k_1 is the wave vector magnitude, given by:

$$k_1 = \frac{\omega}{V_1}, V_1 = \frac{1}{\sqrt{\rho_1 \chi_1}}$$

with ρ_1 and χ_1 representing the density and adiabatic compressibility coefficient of the fluid, respectively.

The pressure wave generates a reflected wave propagating in the opposite direction and a transmitted wave traveling into fluid (2). These waves are sinusoidal and progressive.

1. Provide, with justification, the real expressions of the acoustic overpressures associated with the reflected and transmitted waves.
2. Using Euler's equation, compute the particle velocities$\dot{u}_i$, $\dot{u}_r$, and $\dot{u}_t$ for the incident, reflected, and transmitted waves.
3. The acoustic impedance of a fluid at a point x and time t is given by: $Z = \frac{p(x,t)}{S\dot{u}(x,t)}$

   a. Using the continuity conditions at the interface, derive the equations relating the amplitudes of the incident, reflected, and transmitted waves.
   b. Deduce the reflection and transmission coefficients R_p and T_p in terms of the impedances Z_1 and Z_2.
4. The average acoustic power carried by each wave is given by:

5. $\langle P\rangle = S\langle p(x,t)u(x,t)\rangle$

a. Determine the reflection α_R and transmission α_T coefficients related to acoustic power. What can be observed about these coefficients?
b. Compute α_R and α_T for air (Z_1= 450 Rayleigh) and water (Z_2=1.5×10^6 Rayleigh).
c. Repeat the calculation for water and soil (Z_3=4.2×10^6 Rayleigh).

By analyzing the results obtained in b and c, can we determine whether a fish in the water is frightened when a fisherman speaks or when he moves on the shore?

Solution:

1) The incident wave at x=0 is written as:

$$p_i(x,t) = Ae^{j(\omega t - k_1 x)}$$

This wave propagates in medium 1 (x<0) in the direction of increasing x. At x=0, there is an impedance variation, which gives rise to a reflected wave that is regressive in medium 1 and a transmitted wave that is progressive in medium 2. These two waves have the same angular frequency ω\omegaω, with their respective amplitudes denoted as B and C:

$$p_r(x,t) = B\ e^{j(\omega t + k_1 x)}$$

$$p_t(x,t) = C\ e^{j(\omega t - k_2 x)}$$

With:

$$k_1 = \frac{\omega}{V_1}$$

$$k_2 = \frac{\omega}{V_2}$$

2) Using Euler’s equation, we obtain:

$$\dot{u}_I(x,t) = \frac{A}{\rho_1 V_1}\ e^{j(\omega t - k_1 x)}$$

$$\dot{u}_r(x,t) = -\frac{B}{\rho_1 V_1}\ e^{j(\omega t + k_1 x)}$$

$$\dot{u}_t(x,t) = \frac{C}{\rho_2 V_2}\ e^{j(\omega t - k_1 x)}$$

3) Reflection and transmission coefficients

a. Using continuity conditions, we have:

$$p_i(0,t) + p_r(0,t) = p_t(0,t) \Rightarrow A + B = C$$

$$\dot{u}_i(0,t) + \dot{u}_r(0,t) = \dot{u}_t(0,t) \Rightarrow \frac{1}{\rho_1 V_1}(A - B) = \frac{C}{\rho_2 V_2}$$

b. From the continuity equations, we find:

$$R_p = \frac{Z_1 - Z_2}{Z_1 + Z_2}, T_p = \frac{2Z_2}{Z_1 + Z_2}$$

4) Using the definition of power, we find:

$$\langle p_i \rangle = \frac{1}{2}\frac{SA^2}{\rho_1 V_1} = \frac{A^2}{2Z_1}$$

where $\langle p_i \rangle$, $\langle p_r \rangle$ and $\langle p_t \rangle$ are the average powers of the incident, reflected, and transmitted waves, respectively.

a. Power reflection and transmission coefficients:

$$\alpha_R = \frac{\langle P_r \rangle}{\langle P_i \rangle} = R_p^2 = \left(\frac{Z_2 - Z_1}{Z_1 + Z_2}\right)^2$$

$$\alpha_T = \frac{\langle P_t \rangle}{\langle P_i \rangle} = \frac{Z_1}{Z_2}T_p^2 = \frac{4Z_1 Z_2}{(Z_1 + Z_2)^2}$$

We observe that α_R+α_T=1, which expresses the conservation of power in the conservative case.

b. Case of air-to-water reflection:

We find:

$$\alpha_R = 0.99 \text{ and } \alpha_T = 0.01$$

The reflection is almost total.

c. Case of water-to-earth reflection:

We find:

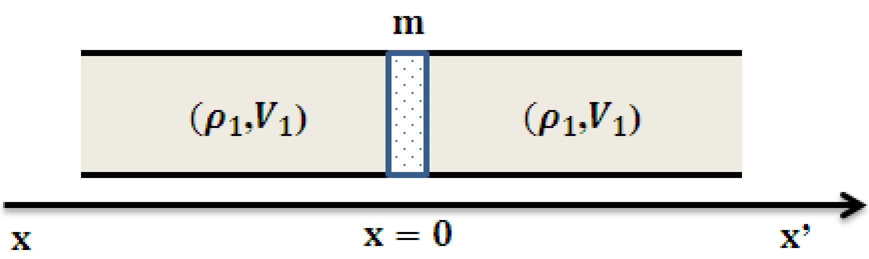
$$\alpha_R = 0.22 \text{ and } \alpha_T = 0.78$$

Based on the previous results (questions 4.b. and 4.c.), we conclude that a fish is more likely to be frightened when the fisherman moves on the shore than when he speaks.

Exercise 2

We consider a cylindrical pipe of infinite length and cross-section S, containing a fluid of volumetric mass ρ_1. At x=0, there is a mass m, made of a rigid cylinder with cross-section S and negligible thickness.

- This mass can slide without friction inside the pipe.
- A plane acoustic wave of pulsation ω and amplitude p_0 propagates along the increasing x direction with speed V.



1. Derive the expression for the pressure $p_1(x,t)$ for x<0 and $p_2(x,t)$ for x>0 in terms of the problem's given parameters and the reflection and transmission coefficients R and T at x=0. Deduce the expressions for the particle velocity $\dot{u}_1(x,t)$ and $\dot{u}_2(x,t)$ in each region x<0 and x>0.
2. Write the fundamental dynamic equation at x=0.
3. Express the continuity condition for the particle velocity at x=0 and derive the first equation relating R and T. Using the fundamental dynamic equation applied to mass m, establish the second equation relating R and T. Show that the reflection coefficient at x=0 is:

$$R = \frac{jm\omega}{\alpha + jm\omega}$$

Express α in terms of the problem's parameters.

Solution:

1. Pressure and Velocity Expressions:

- The pressures are given by:

$p_1(x,t) = A\left[e^{j(\omega t-kx)} + R\, e^{j(\omega t+kx)}\right]$ and $p_2(x,t) = A\, T e^{j(\omega t-kx)}$

(Since the medium is the same on both sides of the mass, the velocities are the same.)

- The velocities are:

$$\dot{u}_1(x,t) = \frac{1}{\rho V}\left[e^{j(\omega t-kx)} - R\, e^{j(\omega t+kx)}\right]$$

$$\dot{u}_2(x,t) = \frac{1}{\rho V} A\, T e^{j(\omega t-kx)}$$

2. Fundamental Dynamic Equation:

- The force balance equation (FDE) is:

 $Sp_1|_{x=0} - Sp_2|_{x=0} = m\,\ddot{u}$

 which simplifies to:

$$Sp_i(1+R-T) = jm\omega\chi VTp_i$$

 Note: Since the displacement of particles is continuous at $x=0$, the acceleration is also continuous.

3. Continuity of Velocities:

- Velocity continuity at x=0 gives:

$$1-R=T$$

- From this, we deduce the reflection coefficient:

$$R = \frac{jm\omega/V}{2S+jm\omega/V} = \frac{jm\omega}{\alpha+jm\omega}$$

$$\alpha = \frac{2S}{\chi V}$$

Exercise 3

Part A: Tube 2 terminated by its characteristic impedance Z_{c2}

A semi-infinite tube (Tube 1) of cross-sectional area S_1 is connected at x=0 to a second pipe (Tube 2), which has a length L, a cross-sectional area S_2, and is terminated at x =L by its characteristic acoustic impedance Z_{c2} (see Figure 1.a). Both tubes are filled with the same fluid of density ρ, in which acoustic waves propagate at speed c. A sinusoidal incident pressure wave with angular frequency ω and complex amplitude A_1 propagates in tube 1 toward the junction at x=0. Upon reaching this point, the incident wave generates a reflected wave of complex amplitude B_1 in tube 1 and a transmitted wave of complex amplitude A_2 in tube 2.

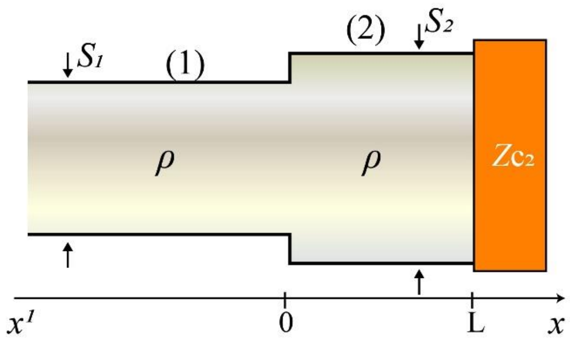


Figure 1.a

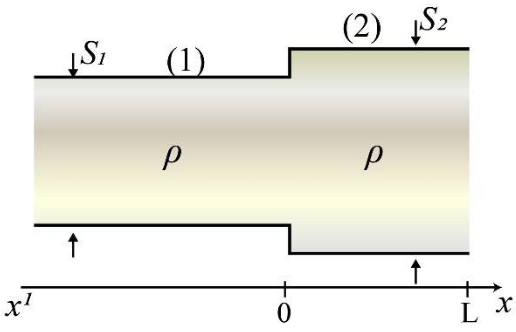


Figure 1.b

1. Write the expressions of the pressure waves p_1(x,t) and p_2(x,t) in tubes 1 and 2, respectively. Deduce the corresponding expressions for the particle velocity waves $\dot{u}_1(x,t)$ et $\dot{u}_2(x,t)$.

2. By applying the continuity conditions at x=0, determine the reflection coefficient r_0 and the transmission coefficient t_0 in pressure at the junction (x =0). Perform the numerical application in the case where S_2= $2S_1$.

3. Assuming the pipe cross-sections satisfy S_2= $2S_1$,

   (a) show that the acoustic pressure in tube 1 can be written in the form:

$p_1(x,t)$ = P (x)$e^{j(\omega t-kx)}$ where P(x) is a complex function to be determined.

(b) Determine the positions and real amplitude P_{max} of the pressure maxima, as well as the positions and real amplitude P_{min} of the pressure minima.

4. Deduce the standing wave ratio (SWR), defined as the ratio: $\tau = \frac{P_{max}}{P_{min}}$.

Part B: Tube 2 open to the exterior

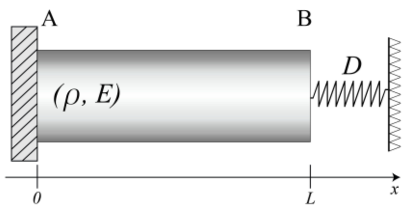
Pipe 2 is now open to the outside at x=L (see Figure 1.b).

1- Write the expressions for the pressure waves $p_1(x,t)$ and $p_2(x,t)$ in tubes 1 and 2, respectively. Then deduce the corresponding expressions for the particle velocity waves $\dot{u}_1(x,t)$ *and* $\dot{u}_2(x,t)$.

1. Using the continuity conditions at x=0 and x=L (continuity of pressure and particle velocity), determine the pressure reflection coefficient r_0' at x = 0.
2. Find the minimum length L_{min} of tube 2 such that there is a pressure node at x=0.

<u>**Exercice 4**</u>

A solid bar of length L, cross-section S, density ρ and rigidity coefficient E is considered. One end (A), abscissa x = 0, is recessed. The other end (B), abscissa x = L, is connected to a spring of stiffness D, the other end of which is connected to the wall. V is the velocity of longitudinal wave propagation at any point x on the bar.



The displacement at any point x of the solid bar is written:

$$u(x,t) = Ae^{j(\omega t-kx)} + B\, e^{j(\omega t+kx)}$$

1. Using the condition at x = 0, show that standing waves are established, described by a longitudinal displacement of the form $u(x,t) = f(x)e^{j\omega t}$ where $f(x)$ is a function to be explained.

2. Using the fundamental principle of dynamics at $x = L$, deduce the proper pulsation equation.

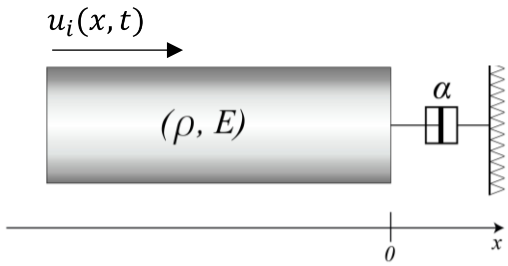
3. What do we observe:

a- When D = 0

b- When D is infinite.

Exercice 5

Consider a semi-infinite solid bar of density ρ and coefficient of rigidity E, whose end $x = 0$ is connected to a damper with. An incident wave coming from the left gives rise to a reflected wave at $x = 0$. we assume $Z_c \geq \alpha$, where Z_c specify the characteristic impedance of the solid rod.



1- a- Provide the reflection coefficient r for displacement at $x = 0$ as function of α *and* Z_c.

b- Montrer que l'onde résultante de déplacement se mettre sous la forme :

$$u(x,t) = U(x)e^{j(\omega t - kx)}$$

Où $U(x)$ est une fonction complexe à déterminer en fonction de r et des données du problème ;

c- Déterminer les positions et l'amplitude U_{min} des minimas d'amplitude (nœuds) ainsi que les positions et l'amplitude des maximas d'amplitude (ventres). En déduire le taux d'ondes stationnaires T.O.S défini par $\tau = \frac{U_{max}}{U_{min}}$.

2- On suppose maintenant que maintenant que $\alpha = \infty$.

a- Que devient l'expression de $u(x,t)$?

b- Déterminer les points qui vibrent avec une amplitude maximale (ventres) et minimale (nœuds).

3- Quelle est la nature de $u(x,t)$ dans le cas où $\alpha = Z_c$?

Solution:

1-a. The displacement reflection coefficient is expressed as: r = $\frac{z_c-\alpha}{z_c+\alpha}$

b. The resulting wave in terms of displacement is given by:

$$\boldsymbol{y}(\mathrm{x,t}) = y_0\left[e^{j(\omega t-kx)} + re^{j(\omega t-kx)}\right] = y_0(1+re^{2jkx})e^{j(\omega t-kx)}$$

Where $Y(x) = y_0(1+re^{2jkx})$

c. The magnitude of the displacement amplitude is given by:

$|Y(x)| = y_0\left|1+re^{2jkx}\right| = y_0\sqrt{1+r^2+2r\cos(2kx)}$

- The amplitude reaches its maxima when $\cos 2kx = +1$ so:

  $|Y(x)| = Y_{max} = 1+r$

Let: $\cos 2kx = +1 \Rightarrow 2kx_{max} = 2n\pi \Rightarrow x_{max} = n\frac{\lambda}{2}\ (n = 0,-1,-2\ \ldots.)$

- The amplitude reaches its minima whencos $2kx = -1$ so:

$$|Y(x)| = Y_{min} = 1-r$$

Let: $\cos 2kx = -1 \Rightarrow x_{min} = (2n-1)\frac{\lambda}{4}(n = 0,-1,-2\ \ldots..)$

From this, the Standing Wave Ratio (SWR) is deduced: $\tau = \frac{Y_{max}}{Y_{min}}$

2- a. If $\alpha = 0, r = +1$

$$y(x,t) = y_0\left[e^{j(\omega t-kx)} + e^{j(\omega t+kx)}\right] = 2y_0\cos(kx)e^{j\omega t}$$

b. The antinodes are located at positions where $\cos(kx) = \pm\ 1$, let $kx_{max} = n\pi$.

Where: $x_{max} = \frac{n\lambda}{2}\ (n = 0,-1,-2,\ldots)$

The nodes occur at positions where $\cos(kx) = \pm 0$, let $kx_{min} = \frac{(2n-1)\pi}{2}$

Where: $x_{min} = \frac{(2n-1)\lambda}{4}\ (n = 0,-1,-2,\ldots)$

3- if $\alpha = Z_c, r = 0$:

$$y(x,t) = y_0e^{j(\omega t-kx)}$$

Chapter V: Electromagnetic Waves in Vacuum

V- Introduction

Let us imagine a particle that has an electric charge. Due to its charge, the particle generates an electric field around it. This electric field is static, it does not vary over time, because the charge remains at rest.

Now, let us imagine that we accelerate this charge, making it move. As we accelerate the particle, it will acquire a certain velocity. According to special relativity, this velocity causes the appearance of a magnetic field around it. But electric and magnetic fields are linked. They are two aspects of a single entity, the electromagnetic field. They will therefore interact with each other. In particular, the newly created magnetic field will disturb the electric field, and this disturbance of the electric field will, in turn, disturb the magnetic field.

Gradually, the electric and magnetic fields will influence each other, varying one after the other, disturbance after disturbance. In this way, the motion that we imparted to the particle will propagate throughout space, slowly, spreading at the velocity of light through the electric and magnetic fields. This is the phenomenon we call an electromagnetic wave.

Depending on the acceleration of the particle that generates it, an electromagnetic wave will be more or less energetic. Thus, we can classify waves into several categories of energy, depending on the frequency at which they oscillate.

These include visible light, with the different colors of the rainbow, infrared, ultraviolet, microwaves, X-rays, radio waves, and gamma rays. With current technologies, it is possible to design special cameras and measuring instruments to detect or emit them.

V-1- Maxwell's Equations in Vacuum

In a vacuum, the electromagnetic field obeys Maxwell's equations:

1. Gauss's Law for the Electric Field:

$$\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_0}$$

This equation states that the divergence of the electric field E is proportional to the charge density ρ, meaning that electric field lines originate from positive charges and terminate on negative charges.

2. Gauss's Law for the Magnetic Field (or Maxwell-Thomson Equation):

$$\vec{\nabla}.\vec{B} = 0$$

This equation expresses that there are no magnetic monopoles; in other words, magnetic field lines always form closed loops.

3. Faraday's Law of Induction (Maxwell-Faraday Equation):

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

This law describes how a time-varying magnetic field induces an electric field, forming the basis of electromagnetic induction.

4. Ampère-Maxwell Law:

$$\vec{\nabla} \times \vec{B} = \mu_0 J_c + \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$$

This equation extends Ampère's law by including Maxwell's correction: a changing electric field $\frac{\partial \vec{E}}{\partial t}$ can generate a magnetic field, even in the absence of conduction currents.

These four fundamental equations fully describe the behavior of electromagnetic fields in free space. They demonstrate how electric and magnetic fields are interrelated and how they propagate as electromagnetic waves in a vacuum.

V-2- Nabla Operator

The Nabla operator is defined as:

$$\vec{\nabla} = \frac{\partial}{\partial x}\vec{e}_x + \frac{\partial}{\partial y}\vec{e}_y + \frac{\partial}{\partial z}\vec{e}_z$$

Where $\overrightarrow{(\boldsymbol{e}_x}, \vec{\boldsymbol{e}}_y, \vec{\boldsymbol{e}}_z)$ are the unit vectors of the coordinate system.

V-3- Gradient

The gradient of a scalar function *T(x,y,z)* is a vector defined by:

$$\vec{\nabla}.T = \frac{\partial}{\partial x}T\,\vec{e}_x + \frac{\partial}{\partial y}T\,\vec{e}_y + \frac{\partial}{\partial z}T\,\vec{e}_z$$

This vector points in the direction of the greatest rate of increase of the function TTT and its magnitude represents the rate of change in that direction.

V-4- Divergence

The divergence of a vector function $\vec{V}$ with components V_x(x,y,z), V_y(x,y,z) et V_z(x,y,z) is a scalar defined by:

$$\vec{\nabla}.\vec{V} = \frac{\partial}{\partial x}V_x + \frac{\partial}{\partial y}V_y + \frac{\partial}{\partial z}V_z$$

Divergence represents the rate at which a vector field expands or contracts at a given point.

V-5- Curl (Rotationnel)

The curl of a vector function V is defined as:

$$\vec{\nabla} \times \vec{V} = \vec{\nabla} \wedge \vec{V} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

Which expands to:

$$\vec{\nabla} \times \vec{V} = \vec{e}_x\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) - \vec{e}_y\left(\frac{\partial V_z}{\partial x} - \frac{\partial V_x}{\partial z}\right) + \vec{e}_z\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)$$

The curl measures the rotational tendency of a vector field at a point.

V-6- Laplacian

The Laplacian of a scalar function T(x,y,z) is given by:

$$\Delta\, T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \vec{\nabla}\,.(\vec{\nabla}\,.\ T)$$

The Laplacian of a scalar function describes how the function's values diverge from its average in a small neighborhood.

For a vector function $\vec{V}$, the Laplacian is defined as:

$$\Delta\vec{V} = \left(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2}\right)\vec{e}_x + \left(\frac{\partial^2 V_y}{\partial x^2} + \frac{\partial^2 V_y}{\partial y^2} + \frac{\partial^2 V_y}{\partial z^2}\right)\vec{e}_y + \left(\frac{\partial^2 V_z}{\partial x^2} + \frac{\partial^2 V_z}{\partial y^2} + \frac{\partial^2 V_z}{\partial z^2}\right)\vec{e}_z$$

The Laplacian of a vector field is applied component-wise and is useful in fields such as fluid dynamics and electromagnetism.

V-7- Maxwell's Equations in Vacuum

The fundamental constants:

$$\varepsilon_0 = 8.85\ 10^{-12}\ \mathrm{F/m}$$

$$\mu_0 = 4\pi\ 10^{-7}\mathrm{H/m^{-1}}$$

Maxwell's equations in a vacuum, where charge and current densities are zero, are given by:

1. Gauss's Law for the Electric Field:

$$\vec{\nabla}.\vec{E} = 0$$

2. Gauss's Law for the Magnetic Field (or Maxwell-Thomson Equation):

$$\vec{\nabla}.\vec{B} = 0$$

3. Faraday's Law of Induction (Maxwell-Faraday Equation):

$$\vec{\nabla} \times \vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

4. Ampère-Maxwell Law:

$$\vec{\nabla} \times \vec{B} = \mu_0\ J_c + \varepsilon_0\ \mu_0\ \frac{\partial\vec{E}}{\partial t}$$

Where the speed of light in vacuum is given by:

$$C^2 = \frac{1}{\varepsilon_0\ \mu_0}$$

V-8- D'Alembert's Equation or the Wave Equation

We start with the vector identity for the curl of the curl of a vector field and apply it to the electric field $\vec{E}$:

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla}\,(\vec{\nabla}\,.\vec{E}) - \Delta\vec{E}$$

From Maxwell's equations, we know that in free space:

$$\vec{\nabla}\,.\vec{E} = 0$$

Thus, the equation simplifies to:

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\Delta\vec{E}$$

Next, we compute $\vec{\nabla} \times (\vec{\nabla} \times \vec{E})$ using Maxwell's equations. From Faraday's law:

$$(\vec{\nabla} \times \vec{E}) = (-\frac{\partial \vec{B}}{\partial t})$$

Taking the curl of both sides:

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla} \times (-\frac{\partial \vec{B}}{\partial t})$$

Using the time derivative property of the curl:

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\frac{\partial}{\partial t} (\vec{\nabla} \times \vec{B})$$

From Maxwell-Ampère's law:

$$\vec{\nabla} \times \vec{B} = \frac{1}{C^2} \frac{\partial \vec{E}}{\partial t}$$

Substituting this into our equation:

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\frac{1}{C^2} \frac{\partial}{\partial t} (\frac{\partial \vec{E}}{\partial t})$$

which simplifies to:

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\frac{1}{C^2} \frac{\partial^2 \vec{E}}{\partial t^2}$$

From Equations (1) and (2), we obtain the wave equation for the electric field:

$$\Delta\vec{E} - \frac{1}{C^2} \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

This is the D'Alembert wave equation, which describes how electromagnetic waves propagate in free space. The same derivation can be done for the magnetic field $\vec{B}$, leading to a similar wave equation.

V-9- D'Alembert's Equation or the Wave Equation for the Magnetic Field

We start with the vector identity for the curl of the curl of a vector field, applying it to the magnetic field $\vec{B}$:

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \vec{\nabla} (\vec{\nabla} . \vec{B}) - \Delta\vec{B}$$

From Maxwell’s equations in free space, we know that:

$$\vec{\nabla}\,.\,\vec{B} = 0$$

Thus, the equation simplifies to:

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = -\Delta\vec{B}$$

Now, we compute $\vec{\nabla} \times (\vec{\nabla} \times \vec{B})$ using Maxwell’s equations. From Maxwell-Ampère’s law:

$$(\vec{\nabla} \times \vec{B}) = \frac{1}{C^2}\frac{\partial\vec{E}}{\partial t}$$

Using the time derivative property of the curl:

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \frac{1}{C^2}\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{E})$$

From Faraday’s law:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

Substituting this into our equation:

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = -\frac{1}{C^2}\frac{\partial}{\partial t}(\frac{\partial\vec{B}}{\partial t})$$

which simplifies to:

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = -\frac{1}{C^2}\frac{\partial^2\vec{B}}{\partial t^2}$$

From Equations (3) and (4), we obtain the wave equation for the magnetic field:

$$\Delta\vec{B} - \frac{1}{C^2}\frac{\partial 2\vec{B}}{\partial t^2} = 0$$

This equation shows that the magnetic field $\vec{B}$ also satisfies the wave equation, just like the electric field $\vec{E}$. Together, these equations describe how electromagnetic waves propagate in free space.

V-10- D'Alembert’s Equations or the Wave Equations

The wave equations derived for the electric and magnetic fields are:

$$\Delta\vec{E} - \frac{1}{C^2}\frac{\partial^2\vec{E}}{\partial t^2} = 0$$

$$\Delta\vec{B} - \frac{1}{C^2}\frac{\partial^2\vec{B}}{\partial t^2} = 0$$

These equations describe the propagation of electromagnetic waves.

We observe that the propagation speed of both fields is the same and is given by:

$$V = \frac{1}{\sqrt{\mu\varepsilon}}$$

In a vacuum, the speed of light is:

$$V = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = 3\ .10^8\ m/s$$

This result confirms that electromagnetic waves propagate at the speed of light in a vacuum.

V-11- Solution in Sinusoidal Plane Waves

A sinusoidal progressive plane wave solution for the electric field E is given by:

$$\vec{E}\,(\vec{r},t) = \vec{E}_0 e^{j(\omega t - \vec{k}\vec{r})}$$

where:

- $\vec{E}$ is the electric field.
- $\vec{E}_0$ is the amplitude vector and also represents the polarization vector.
- ω is the angular frequency of the wave.
- $\omega t - \vec{k}\vec{r}$ is the instantaneous phase, which depends on time.

V-12- Dispersion Relation

By substituting the plane wave solution into the wave equation:

$$\Delta\vec{E} - \frac{1}{C^2}\frac{\partial^2\vec{E}}{\partial t^2} = 0$$

and using the definition of the vector Laplacian, we find that:

$$k = \frac{\omega}{C}$$

where k is the wavenumber and C is the speed of light.

This equation represents the dispersion relation, which determines the phase velocity of the wave:

$$V_{phase} = \frac{\omega}{k} = C$$

This confirms that electromagnetic waves propagate at the speed of light in a vacuum.

V-13- Transversality of Electromagnetic Waves

1. The Electric Field is Perpendicular to the Propagation Direction

In the absence of free electric charge (ρ=0), Gauss's law states:

$$\vec{\nabla} . \vec{E} = 0$$

For a plane wave solution of the form:

$$\vec{E}(\vec{r}, t) = \vec{E}_0 e^{j(\omega t - \vec{k}\vec{r})}$$

we apply the divergence operator:

$$\text{div}(\vec{E}) = -i\,\vec{k}.\vec{E} = 0$$

which simplifies to:

$$\vec{k}.\vec{E} = 0$$

This result means that the electric field E is perpendicular to the wave vector k, which represents the direction of wave propagation. Thus, the electric field is transverse to the direction of wave propagation.

2. The Magnetic Field is Also Transverse

From Maxwell-Faraday's equation:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Substituting the plane wave solution and solving for B, we obtain:

$$\vec{B} = \frac{\vec{k} \times \vec{E}}{\omega}$$

Since $\vec{E}$, is perpendicular to $\vec{k}$, and $\vec{B}$ is given by a cross product of $\vec{k}$ and $\vec{E}$,, this means that the magnetic field $\vec{B}$ is also perpendicular to the direction of propagation. Thus, electromagnetic waves are transverse waves, with both $\vec{E}$, and $\vec{B}$ perpendicular to the propagation direction $\vec{k}$.

3. Conclusion

- $\vec{E}$ is perpendicular to $\vec{k}$
- $\vec{B}$ is perpendicular to $\vec{B}$,
- $\vec{E}$and $\vec{B}$ are also perpendicular to each other, forming a right-handed orthogonal system where the wave propagates in the direction of $\vec{k}$.

V-14- Orientation and Magnitude of the Magnetic Field in a Plane Wave

1. The Magnetic Field is Perpendicular to the Plane ($\vec{k}$, $\vec{E}$)

From the expression of the magnetic field:

$$\vec{B} = \frac{\vec{k} \times \vec{E}}{\omega}$$

we use the properties of the cross product:

- The result of a cross product $\vec{A}\times\vec{B}$ is a vector perpendicular to both $\vec{A}$ and $\vec{B}$.

Thus, since $\vec{B} = \frac{\vec{k}\times\vec{E}}{\omega}$, we conclude that:

- $\vec{B}$ is perpendicular to both $\vec{B}$and $\vec{E}$.
- This confirms that the magnetic field is also transverse to the propagation direction.

2. The Right-Handed Coordinate System

Since $\vec{B}$ is given by the cross product:

$$\vec{B} \propto \vec{k} \times \vec{E}$$

This means that the three vectors $\vec{k}$, $\vec{E}$, $\vec{B}$ form a right-handed coordinate system (trièdre direct):

- $\vec{E}$ *and* $\vec{B}$ are both perpendicular to $\vec{k}$.
- The direction of $\vec{B}$ follows the right-hand rule: if you point the fingers of your right hand in the direction of $\vec{k}$ and curl them towards $\vec{E}$, your thumb will point in the direction of $\vec{B}$.

Thus, an electromagnetic wave consists of:

- An electric field $\vec{E}$ oscillating in one direction.
- A magnetic field $\vec{B}$ oscillating in a perpendicular direction.
- A wave vector $\vec{k}$ giving the direction of propagation.

3. Magnitude of the Magnetic Field

Using the dispersion relation:

$$\omega = kC$$

we substitute into the magnetic field equation:

$$\|\vec{B}\| = \frac{\|\vec{E}\|}{C}$$

This result shows that the amplitude of the magnetic field is smaller than the electric field by a factor of *C*. Since *C* is a large value ($3 \times 10^8\ m/s$), the magnetic field is typically much weaker than the electric field in magnitude.

V-15- Polarization of Electromagnetic Waves

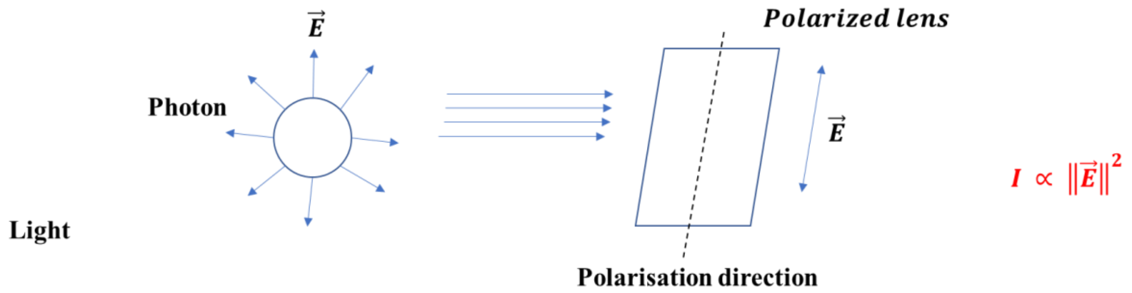
1. Definition of Polarization

Polarization describes the geometric shape traced by the tip of the electric field vector $\vec{E}$ as the wave propagates over time.

In other words, polarization represents the trajectory of $\vec{E}$ in a fixed plane perpendicular to the propagation direction.

2. Example of Polarization Explained Simply

The image illustrates the principle of light polarization through three key stages:



2.1. Unpolarized Light (Natural Light Source)

On the left side of the image, we see light waves emitted by a natural light source (like the sun or a lamp). This light is unpolarized, meaning the electric field vector $\vec{E}$ vibrates in all possible directions perpendicular to the direction of propagation.

The electric field components are randomly oriented, forming a circular pattern around the propagation axis.

2.2. Polarizing Filter (Polarizer)

When the light passes through the polarizing filter:

- The polarizer blocks all electric field components except the ones vibrating in a specific direction.
- Only the electric field components parallel to the polarizer's transmission axis are allowed to pass.

The filter selects one direction of oscillation, transforming unpolarized light into linearly polarized light.

2.3. Polarized Light (After the Filter)

After the polarizer, the electric field oscillates along one fixed direction only. This is called linear polarization.
The light intensity is reduced according to the relation:

$$I \propto \|\vec{E}\|^2$$

2.4. Conclusion

- The polarizer acts like a filter that forces the electric field to oscillate in only one direction.

- This example shows that polarization affects the intensity of light and changes its geometric structure.
- Polarization is an important property in optics, used in sunglasses, cameras, and LCD screens.

3. Definition of Rectilinear Polarization

Rectilinear polarization (or linear polarization) refers to a type of wave polarization where the electric field vector oscillates along a single fixed direction while the wave propagates.

In other words:

- The electric field $\vec{E}$ remains confined to a single plane.
- The magnetic field $\vec{B}$ is perpendicular to both the electric field and the wave propagation direction.

Example 1:

A plane electromagnetic wave propagates in the y direction with an electric field given by:

$$\vec{E} = E_0 \cos(\omega t - ky)\vec{e}_z$$

1. Determine the direction of wave propagation.
2. Verify whether the wave is transversely polarized.
3. Identify the polarization type.

Solution:

Step 1: Identifying the Wave Propagation Direction

The given electric field equation is:

$$\vec{E} = E_0 \cos(\omega t - ky)\vec{e}_z$$

We compare this with the general form of a plane wave:

$$\vec{E}\,(\vec{r}, t) = \vec{E}_0 \cos\left(\omega t - \vec{k}\vec{r}\right)$$

From the term $\omega t - ky$, we see that the wave propagates along the y axis. This means the wave vector is:

$$\vec{k} = k\vec{e}_y$$

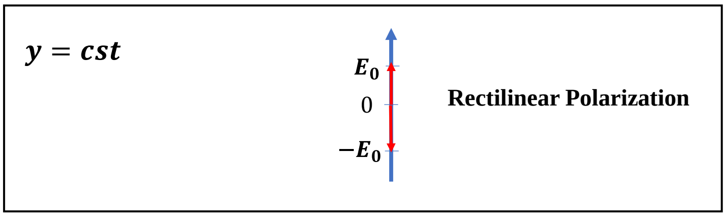
Step 2: Checking if the Wave is Transverse

A wave is transverse if the electric field $\vec{E}$ is perpendicular to the propagation direction.

- The wave vector is along y: $\vec{k} = k\vec{e}_y$.

  The electric field is along z: $\vec{E} = E_0 \cos(\omega t - ky)\vec{e}_z$.

Since $\vec{E}$ is along z and not along y, we conclude that E⊥k.



Thus, the wave is transversely polarized.

Step 3: Identifying the Type of Polarization

Since the electric field oscillates only in one fixed direction (z axis) while the wave moves along the y axis, the polarization is rectilinear (linear).

Conclusion:

- The wave propagates along the y axis.
- The electric field oscillates along the z axis.
- The wave is transverse.
- The polarization is rectilinear (linear).

Final Answer: The given wave represents a linearly polarized transverse electromagnetic wave propagating along the y axis.

4. Polarisation Circulaire

Circular polarization is a type of polarization in which the tip of the electric field vector $\vec{E}$ traces a circle over time in a plane perpendicular to the wave's propagation direction. This means that the components of the electric field along two perpendicular directions (e.g., *x* and *z*) have the same amplitude but are phase-shifted by $\pm\frac{\pi}{2}$.

Example 3: Circular Polarization

Let's analyze the given electric field components:

$$\vec{E}\begin{cases} E_x = \mathrm{A}\cos(\omega t - kz) \\ E_y = A\sin(\omega t - kz) \\ 0 \end{cases}$$

Solution :

Step 1: Understanding the Components

- The electric field has two perpendicular components: E_x along the x axis and E_y along the y axis.
- There is no component along the z axis, meaning the wave is propagating along the z direction.

Step 2: Verifying the Trajectory

To determine the trajectory of the tip of the electric field vector, we square both equations and sum them:

$$E_x^2 + E_y^2 = A^2\cos^2(\omega t - kz) + A^2\sin^2(\omega t - kz)$$

Using the trigonometric identity:

$$\cos^2\theta + \sin^2\theta = 1$$

We obtain:

$$\mathrm{E_x^2 + E_y^2 = A^2}$$

This is the equation of a circle in the $E_x{-}E_y$ plane.

Step 3: Conclusion - Circular Polarization

Since the electric field vector describes a circular motion, the wave is circularly polarized.

The direction of rotation determines whether it is right-hand circular polarization (RHCP) or left-hand circular polarization (LHCP):

- If E_y leads E_x by $+\frac{\pi}{2}$, the wave is left-circularly polarized.

- If E_y lags behind E_x by $-\frac{\pi}{2}$, the wave is right-circularly polarized.

Understanding Circular Polarization Through Step-by-Step Analysis

We analyze the given electric field components:

$$\vec{E}\begin{cases} E_x = A\cos(\alpha) \\ E_y = A\sin(\alpha) \\ 0 \end{cases}$$

where α=ωt−kz represents the wave phase.

Step 1: Checking the Evolution of the Electric Field

Let’s analyze how the electric field vector changes over time by evaluating specific phase values.

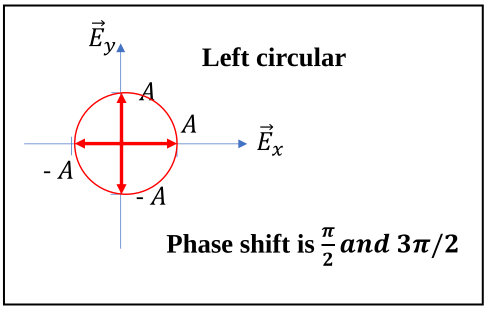
For α=0:

$$E_x = A \; and \; E_y = 0$$

The electric field is entirely along the x axis.

For α = $\frac{\pi}{2}$:

$$E_x = 0 \text{ and } E_y = A$$



For α = π:

$$E_x = -A \text{ and } E_y = 0$$

For α = $\frac{3\pi}{2}$:

$$E_x = 0 \text{ and } E_y = -A$$

Step 2: Observing the Motion

- The electric field starts along the x axis.
- Then it moves towards the positive y axis.
- Then it reverses along the x axis.
- Finally, it moves to the negative y axis before returning to its initial position.

This describes a circular motion of the electric field vector.

5. Definition of Elliptical Polarization

Elliptical polarization occurs when the tip of the electric field vector traces an ellipse in the plane perpendicular to the direction of wave propagation.

- Why does this happen?
  - The electric field has two perpendicular components (e.g., E_x and E_y).
  - These components have different amplitudes and/or a phase difference that is neither 0° nor 180°.
- Key Characteristics:
  1. The electric field rotates as the wave propagates.
  2. The shape of the trajectory is an ellipse (instead of a straight line or a circle).
  3. The rotation can be clockwise (right-handed) or counterclockwise (left-handed), depending on the phase relationship.

Elliptical polarization is a general case of wave polarization that includes linear and circular polarization as special cases.

- If the ellipse becomes a straight line → Linear polarization.
- If the ellipse becomes a perfect circle → Circular polarization.

V-16- Electromagnetic Energy and Poynting Vector

1. Electromagnetic Energy Density

Electromagnetic waves carry energy, which is stored in both the electric field ($\vec{E}$) and the magnetic field ($\vec{B}$). The energy density (energy per unit volume) is given by:

$$\omega_{EM} = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{B^2}{\mu_0}\right)$$

- The first term ($\frac{1}{2}\ \varepsilon_0 E^2$) represents the electric energy density.
- The second term ($\frac{1}{2}\ \frac{B^2}{\mu_0}$) represents the magnetic energy density.

In simple terms: The total energy in an electromagnetic wave is the sum of the energy stored in the electric field and the energy stored in the magnetic field.

2. Total Electromagnetic Energy in a Region

Now, let's consider a certain volume (τ) enclosed by a surface S. The total electromagnetic energy contained in this volume is found by integrating the energy density over the entire volume:

$$W = \iiint_{\tau} \omega_{EM} \, d_{\tau}$$

- Energy in an EM wave is spread throughout space and depends on both $\vec{E}$ and $\vec{B}$.
- By integrating the energy density over a given volume, we find the total energy contained in that region.

3. Instantaneous Power and Electromagnetic Energy Variation

1. Power and Energy Change in a Volume

When studying electromagnetic waves, it's important to understand how energy changes over time in a given volume *τ*.

- During a small time interval *dt*, the increase in energy in the volume is *dW*.
- The instantaneous power (energy per unit time) acquired by this volume is:

$$p' = \frac{dW}{dt} = \iiint_{\tau} \omega_{EM} \, d_{\tau}$$

- The power p′ represents how fast the total electromagnetic energy inside the volume is changing.
- If energy enters or leaves the region, p′ tells us how much energy is gained or lost per second.

4. Connection with Maxwell's Equations

We can analyze how energy flows in a system using Maxwell's equations. Two key equations describe the behavior of electric and magnetic fields:

$$\vec{\nabla} \times \vec{E} = - \frac{\partial \vec{B}}{\partial t}$$

Faraday's Law of Induction (describes how changing magnetic fields create electric fields):

$$\vec{\nabla} \times (\frac{\vec{B}}{\mu_0}) = \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

This tells us that a time-varying electric field generates a circulating magnetic field, even in the absence of electric currents.

5- Key takeaway for students:

- The instantaneous power p″ tells us how fast the total energy in a given volume is changing.
- Maxwell's equations show how electric and magnetic fields interact to transfer energy.
- These interactions explain how electromagnetic waves carry energy through space.

We are now deriving an important result in electromagnetism: the conservation of electromagnetic energy, which leads to Poynting's theorem.

6. Expression for Power Change in a Volume

We previously found that the instantaneous power acquired by a volume τ is:

$$\frac{\mathrm{dW}}{\mathrm{dt}} = \iiint_{\tau} \omega_{EM}\, d_{\tau}$$

We now express this change using Maxwell's equations.

From Maxwell's equations, we derived:

$$\frac{\mathrm{dW}}{\mathrm{dt}} = \vec{\mathrm{E}} \,.\left[\vec{\nabla} \times \left(\frac{\vec{\mathrm{B}}}{\mu_0}\right)\right] - \frac{\vec{\mathrm{B}}}{\mu_0}\; \vec{\nabla} \times \vec{\mathrm{E}}$$

Using the Vector Identity, a key vector identity states:

$$\vec{\nabla}.\left[\vec{\mathrm{F}} \times \vec{\mathrm{G}}\right] = \vec{\mathrm{G}}\,.\,\vec{\nabla} \times\; \vec{\mathrm{F}} - \vec{\mathrm{F}}\,.\,\vec{\nabla} \times \vec{\mathrm{G}}$$

Applying this identity to our equation, we get:

$$\vec{E}\,.\left[\vec{\nabla} \times \left(\frac{\vec{B}}{\mu_0}\right)\right] = \;\frac{\vec{B}}{\mu_0} \vec{\nabla} \times \vec{E} - \vec{E}\,.\vec{\nabla} \times \frac{\vec{B}}{\mu_0}$$

Substituting this back, we find:

$$\frac{dW}{dt} = -\vec{\nabla}.\left[\vec{E}\times\left(\frac{\vec{B}}{\mu_0}\right)\right]$$

The power is then given by:

$$p' = \frac{dW}{dt} = -\iiint_{\tau}\vec{\nabla}.\left[\vec{E}\times\left(\frac{\vec{B}}{\mu_0}\right)\right]d\tau$$

The instantaneous electromagnetic power p lost by the volume τ is:

$$p = \iiint_{\tau}\vec{\nabla}.\left[\vec{E}\times\left(\frac{\vec{B}}{\mu_0}\right)\right]d\tau$$

According to Gauss-Ostrogradsky's theorem:

$$p = \iint_{(s)}\left[\vec{E}\times\left(\frac{\vec{B}}{\mu_0}\right)\right]d\vec{S} = \iint_{(s)}\vec{R}\,d\vec{S}$$

which implies that:

$$\vec{R} = \vec{E}\times\frac{\vec{B}}{\mu_0}$$

The vector $\vec{R}$ is called the Poynting vector. Its direction at each point indicates the direction of energy flow, and its flux through a surface corresponds to the instantaneous electromagnetic power radiated by that surface.

The instantaneous power p_u passing through a unit surface SSS perpendicular to the direction of propagation is given by:

$$p_u = \iint_{(s)}\vec{R}\,d\vec{S} = \iint_{(s)}\|\vec{R}\|\,\|d\vec{S}\|$$

Since *R* is uniform over the surface, we get:

$$p_u = \|\vec{R}\|\iint_{(s)}\|d\vec{S}\| = \|\vec{R}\|\,S$$

The average power crossing the surface *S* is therefore:

$$\langle p_u\rangle = \sqrt{\frac{\varepsilon_0}{\mu_0}}\frac{E_0^2}{2} = \frac{E_0^2}{2Z_0}$$

where Z_0 is the impedance of free space.

Chapter IV: Electromagnetic Waves in a Plasma, Conductors, and Perfect Dielectrics

IV-1- Electromagnetic Waves in a Plasma

IV-1-1- Definition

A plasma is a gas that is fully or partially ionized, meaning that the atoms or molecules in the gas carry an individual charge. However, the medium remains globally neutral because the number of positive and negative charges is equal.

Plasma is a dilute medium, with large distances between atoms. Since ions are much heavier compared to electrons, they are almost immobile and do not significantly contribute to wave propagation in the plasma.

IV-1-2- History

- 1902: To explain how Marconi's radio waves could propagate beyond the curvature of the Earth, Heaviside and Kennelly hypothesized the existence of a reflective layer in the atmosphere.
- 1925: This layer was experimentally confirmed by Appleton, demonstrating the presence of what is now known as the ionosphere.

IV-1-3- Maxwell's Equations in Plasma

In a medium with permittivity ε_0 and permeability μ_0, Maxwell's equations take the following form:

3.1. Gauss's Theorem for the Electric Field

$$\vec{\nabla}.\vec{E} = 0$$

3.2. Gauss's Theorem for the Magnetic Field (also known as Maxwell-Thomson Equation)

$$\vec{\nabla}.\vec{B} = 0$$

3.3. Faraday's Law (Maxwell-Faraday Equation:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

3.4. Ampère's Law (Maxwell-Ampère Theorem):

$$\vec{\nabla} \times \vec{B} = \mu_0 J_c + \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$$

These equations describe the behavior of electromagnetic fields in a plasma, where the interaction between charged particles and electromagnetic waves plays a crucial role.

IV-1-3- Plasma Conductivity

When a plasma is subjected to an electromagnetic field ($\vec{E}$,$\vec{B}$) and considering non-relativistic charged particles, the equation of motion for a particle of mass m_p and charge q_p is given by:

$$m_p \frac{d\overrightarrow{V_p}}{dt} = q_p \vec{E}$$

For a wave characterized by a one-dimensional propagation function of the form $e^{j(\omega t - \vec{k}\vec{r})}$, we assume the same form for the velocity of the charged particle. This leads to the relation between velocity and the electric field:

$$j\, m_p \omega \overrightarrow{V_p} = q_p \vec{E}$$

Moreover, the current density is given by Ohm's microscopic law:

$$\vec{J} = N\, q_p \overrightarrow{V_p} = \gamma \vec{E}$$

From the previous equations, the conductivity γ in a plasma is:

$$\gamma = -j\, \frac{N\, q_p^2}{m_p \omega}$$

γ represents the plasma conductivity,

N is the number of particles per unit volume,

$\vec{V_p}$ is the velocity of the particle.

This expression highlights that the plasma conductivity is complex and frequency-dependent, which influences how electromagnetic waves propagate within the plasma medium.

IV-1-4- Dispersion Relation in Plasma

Maxwell's equations allow us to derive the dispersion relation for wave propagation in a plasma:

$$k^2 = \frac{\omega^2 - \omega_p^2}{c^2}$$

where c is the speed of light in vacuum, given by:

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$$

and ω_p is the plasma frequency (also called the cutoff frequency), defined as:

$$\omega_p = \sqrt{\frac{N\, q_p^2}{m_p \varepsilon_0}}$$

This plasma frequency separates the frequency domain into two regions:

5.1. Region 1: ω<ω_p (Evanescent Waves)

For frequencies lower than the plasma frequency, the wave vector k is imaginary:

$$k = \pm j \sqrt{\frac{\omega_p^2 - \omega^2}{c^2}}$$

This means that wave propagation is not possible; instead, the wave becomes evanescent and undergoes exponential decay.

5.2. Region 2: ω>ω_p (Wave Propagation)

For frequencies higher than the plasma frequency, the wave vector is real:

$$k = \sqrt{\frac{\omega_p^2 - \omega^2}{c^2}}$$

In this case, the wave can propagate through the plasma, such as in the ionosphere.

IV-1-5- Wave Properties in Plasma

- **Phase Velocity**:

$$V_\varphi = \frac{\omega}{k} = \frac{c}{\sqrt{1 - \frac{\omega_p^2}{\omega^2}}}$$

- **Group Velocity**:

$$V_g = \frac{d\omega}{dk} = c\sqrt{1 - \frac{\omega_p^2}{\omega^2}}$$

- **Refractive Index**:

$$\frac{c}{V_\varphi} = \sqrt{1 - \frac{\omega_p^2}{\omega^2}}$$

This dispersion relation explains the behavior of electromagnetic waves in a plasma and is crucial in understanding wave propagation in the ionosphere and radio communication.

IV-2- Electromagnetic Waves Conductors

IV-2-1- Maxwell's Equations in Neutral Conductors

In a neutral conductor, Maxwell's equations take the following form:

1.1. Gauss's Theorem for the Electric Field

$$\vec{\nabla}.\vec{E} = 0$$

1.2. Gauss's Theorem for the Magnetic Field (also known as Maxwell-Thomson Equation)

$$\vec{\nabla}.\vec{B} = 0$$

1.3. Faraday's Law (Maxwell-Faraday Equation:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

1.4. Ampère's Law (Maxwell-Ampère Theorem):

$$\vec{\nabla} \times \vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t}$$

Using the relation $\vec{H} = \frac{1}{\mu}\vec{B}$ and $D = \varepsilon\vec{E}$:

$$\vec{\nabla} \times \vec{B} = \mu(\vec{J} + \varepsilon\frac{\partial \vec{E}}{\partial t})$$

IV-2-2- Ohm's Law in Conductors

The current density $\vec{j}$ is given by Ohm's microscopic law:

$$\vec{j} = \gamma\vec{E}$$

where γ is the conductivity of the conductor.

IV-2-3- Electrical Conductivity of Metals

The table below presents the electrical conductivity of common metals in units of **(10^6 Sm^{-1})**:

Métaux	Conductivité électrique (10^6 Sm $^{-1}$)
Argent	**61.1**
Cuivre	**58.5**
Aluminium	**36.9**
Or	**44.2**
Fer	**10.1**

Analysis and Interpretation:

- **Silver (Ag)** is the best electrical conductor among common metals due to its crystalline structure and low resistivity.
- **Copper (Cu)** is nearly as conductive as silver but is more widely used due to its lower cost and high ductility.
- **Gold (Au)** is less conductive than copper but is used in applications where corrosion resistance is essential (e.g., electrical contacts).
- **Aluminum (Al)** is lighter than copper and is commonly used in high-voltage power lines despite having lower conductivity.
- **Iron (Fe)** has significantly lower conductivity and is rarely used for electrical transmission.

These values are crucial for applications in electronics, electrical engineering, and electromagnetic wave transmission in conductors.

IV-2-4- Wave Propagation in Conductors and Dispersion Relation

We consider the electric field $\vec{E}$ as a sinusoidal plane wave:

$$\vec{E} = \vec{E}_0 e^{j(\omega t - \vec{k}.\vec{r})}$$

Applying this to Maxwell's equations, we obtain:

1. Gauss's Theorem for the Electric Field:

$$-j\,\vec{k}\,.\,\vec{E} = 0$$

2. Gauss's Theorem for the Magnetic Field:

$$-j\,\vec{k}.\vec{B} = 0$$

3. Maxwell-Faraday Equation:

$$-j\,\vec{k} \times \vec{E} = -\,j\,\omega\,\vec{B}$$

4. Maxwell-Ampère Equation:

$$-j\,\vec{k} \times \vec{B} = (\mu\,\gamma + j\,\omega\,\mu\,\varepsilon)\,\vec{E}$$

Taking the curl of the curl of $\vec{E}$:

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla}\left(\vec{\nabla}\,.\,\vec{E}\right) - \Delta\vec{E} = -\Delta\vec{E}$$

From Maxwell's equations, we obtain:

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla} \times \left(-\frac{\partial\vec{B}}{\partial t}\right) = -\frac{\partial}{\partial t}\,(\vec{\nabla} \times \vec{B}) = -\frac{\partial}{\partial t}\,\mu\left(\gamma\vec{E} + \varepsilon\frac{\partial\vec{E}}{\partial t}\right)$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \mu\,\left(j\omega\gamma + \omega^2\varepsilon\right)\vec{E} \ldots\ldots\ldots(2)$$

By comparing the equations, we derive the dispersion relation:

$$k^2 = \mu\varepsilon\left(1 - j\frac{\gamma}{\omega\varepsilon}\right)\omega^2$$

Since k is complex, in the case of good conductors where $\frac{\boldsymbol{\gamma}}{\boldsymbol{\varepsilon}} \gg \mathbf{1}$, we approximate:

$$\mathbf{k} = \sqrt{\frac{\boldsymbol{\mu\gamma\omega}}{\mathbf{2}}}\,(\mathbf{1} - \boldsymbol{j})$$

The propagation constant k′ and attenuation constant k′′ are equal:

$$k' = k'' = \sqrt{\frac{\mu\gamma\omega}{2}}$$

For a progressive sinusoidal plane wave traveling along **z**, we express the field as:

$$\vec{E} = \vec{E}_0\,e^{j(\omega t - kz)}$$

Expanding $k = k' - jk''$ **:**

$$\vec{E} = \vec{E}_0\, e^{j(\omega t - kz)} = \vec{E}_0\, e^{j[\omega t - (k' - jk'')z]} = \vec{E}_0\, e^{-k''z} e^{j(\omega t - k'z)}$$

This represents a progressive (k′>0) and attenuated (k″>0) wave.

IV-2-5- Phase and Group Velocity Calculation

From k′, we compute:

- **Phase velocity**:

$$V_\varphi = \frac{\omega}{k'} = \sqrt{\frac{2\omega}{\mu\gamma}}$$

- **Group velocity**:

$$V_g = \frac{d\omega}{dk'} = 2\sqrt{\frac{2\omega}{\mu\gamma}}$$

Both velocities increase with frequency ω, which is a key characteristic of wave propagation in conductive media.

IV-3- Electromagnetic Waves in perfect Conductors

IV-3- 1- Skin Depth and Perfect Conductors

The skin depth δ represents the distance over which the amplitude of the electric field $\vec{E}$ (and also the magnetic field $\vec{B}$ is reduced by a factor of **e**.

For copper, the skin depth is determined using the imaginary part of the wave vector k″:

$$\delta(m) = \frac{1}{k''} = \sqrt{\frac{2}{\omega\gamma\mu}} = \frac{1}{\sqrt{\pi f \mu\gamma}}$$

where:

- γ is the conductivity of the material,
- μ is the permeability,
- f is the frequency of the wave.

The following table provides typical values of the skin depth for copper at different frequencies:

Frequency	Skin depth ($\boldsymbol{\delta}$)
50 Hz	**9.38 mm**
10 kHz	**0.66 mm**
100 kHz	**0.21 mm**
1MHz	**66 µm**
1GHz	**2.1 µm**

IV-3- 2- Perfect Conductor Limit

A perfect conductor corresponds to the limit where the conductivity γ is infinite.

Since the current density $\vec{j}$ must remain finite, Ohm's law:

$$\vec{j} = \gamma\vec{E}$$

implies that the electric field inside the conductor must be zero:

$$\vec{E} = 0$$

This leads to the following properties:

- **No charge accumulation**:

$$\rho = \varepsilon \,\mathrm{div}\,\vec{E} = 0$$

- **No magnetic field inside**:
  From Maxwell-Faraday's equation:

$$\frac{\partial\vec{B}}{\partial t} = -(\vec{\nabla} \times \vec{E}) = 0 \Longrightarrow \vec{B} = 0$$

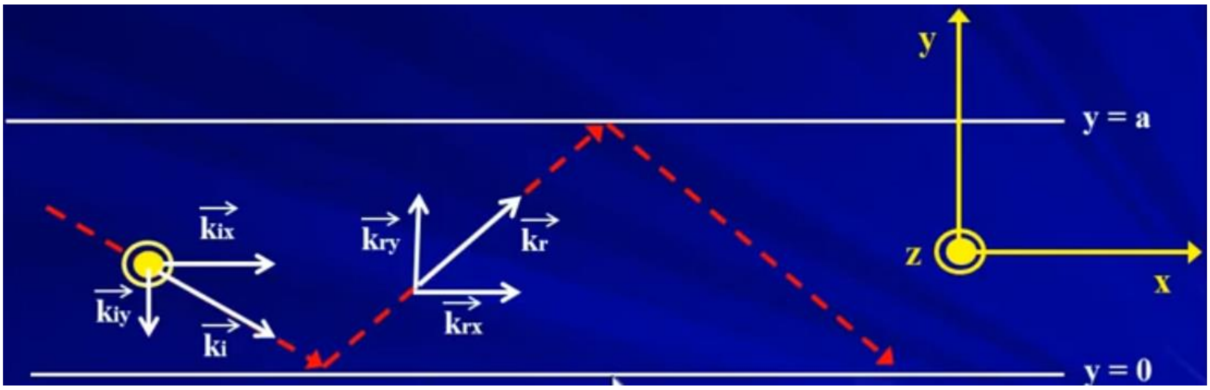
IV-3- 3- Electromagnetic Shielding

As a result, a perfect conductor acts as a shield against electromagnetic fields $\vec{E}$ and $\vec{B}$, preventing any penetration of the wave inside the material.

Chapter VII: Guided Propagation Between Two Parallel Metallic Planes – Application to the Infinite Rectangular Waveguide

VI-1-Introduction

An electromagnetic wave with angular frequency can, under certain conditions, propagate between two parallel conducting planes. We assume these planes to be perfectly conducting and infinitely extended along the direction Ox. One of the planes lies xOy on the plane, while the other is defined by the equation $= a$.



$$\vec{k}_i \begin{cases} k_{ix} = k\sin\theta \\ k_{iy} = -\,k\cos\theta \\ k_{iz} = 0 \end{cases} \qquad \vec{k}_r \begin{cases} k_{rx} = k\sin\theta \\ k_{ry} = -\,k\cos\theta \\ k_{iz} = 0 \end{cases}$$

There are different propagation modes:

- Transverse Electric (TE) mode, where the electric field is transverse.
- Transverse Magnetic (TM) mode, where the magnetic field is transverse.

In this study, we focus on modes where the electric field is polarized along the direction.

To simplify notation, we define:

$$k\sin\theta = \alpha \quad \text{and} \quad k\cos\theta = \beta$$

VI-2- Determination of the Electric Field:

The incident and reflected electric fields can be written as:

$$\vec{E}_i = E_{0i} e^{i(\omega t - \alpha x + \beta y)} \vec{e}_z$$

$$\vec{E}_r = \vec{E}_{0r} e^{i(\omega t - \alpha x - \beta y)}$$

The continuity of the electric field at y=0 gives:

$$E_{0i}\,\vec{e}_z + \vec{E}_{0r} = 0$$

Thus, the reflected field is:

$$\vec{E}_r = -\vec{E}_{0i}e^{i(\omega t-\alpha x-\beta y)}\vec{e}_z$$

Finally, in the waveguide, the resulting electric field is:

$$\vec{E} = \vec{E}_i + \vec{E}_r = A\sin\beta y\, e^{i(\omega t-\alpha x)}\vec{e}_z$$

The continuity of the electric field at y=a gives:

$$\sin\beta a = 0$$

This implies:

$$\beta = n\,\frac{\pi}{a}$$

Where n is a non-zero integer.

Summary:

$$\vec{E} = A\sin\beta y\, e^{i(\omega t-\alpha x)}\vec{e}_z,\ \beta = n\,\frac{\pi}{a}$$

VI-3- Dispersion Relation

The electric field satisfies the wave equation:

$$\Delta\vec{E} - \frac{1}{c^2}\,\frac{\partial 2\vec{E}}{\partial 2t} = 0$$

For a sinusoidal wave, the dispersion relation in the waveguide is:

$$\alpha_n^2 = \left(\frac{\omega}{c}\right)^2 - \left(\frac{n\pi}{a}\right)^2$$

Defining $= \boldsymbol{k}$, we obtain:

$$k_n^2 = \left(\frac{\omega}{c}\right)^2 - \left(\frac{n\pi}{a}\right)^2$$

Or equivalently:

$$\alpha_n^2 = \frac{1}{c^2}\left(\omega^2 - \omega_{cn}^2\right)$$

where the cutoff frequency is:

$$\omega_{cn} = \frac{\pi c}{a}$$

For the first mode:

$$k_1^2 = \frac{1}{c^2}(\omega^2 - \omega_c^2), \text{ with } \omega_c = \frac{\pi c}{a}$$

VI-4- Wave Propagation Analysis

- If $\omega > \omega_c$, the wave propagates with:

$$V_\varphi = \frac{\omega}{k_n} = \frac{c}{\sqrt{1 - \frac{\omega_{cn}^2}{\omega^2}}}$$

$$V_g = \frac{d\omega}{dk_n} = c\sqrt{1 - \frac{\omega_{cn}^2}{\omega^2}}$$

- If $\omega < \omega_c$, the wave is evanescent and does not propagate:

$$\vec{E} = A \sin\frac{\pi y}{a} e^{-\frac{x}{\delta}} e^{i\omega t}\vec{e}_z$$

$$k_n = \pm i\sqrt{\left(\frac{\pi}{a}\right)^2 - \left(\frac{\omega}{c}\right)^2} = \pm\frac{i}{\delta}$$

VI-5- Magnetic Field Calculation

Using Maxwell's equation:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

We obtain:

$$\vec{B} = \frac{A}{\omega}[i\,\beta\cos\beta y\,\vec{e}_x - k\sin\beta y\,\vec{e}_y]\,e^{i(\omega t - kx)}$$

Since $\boldsymbol{B_x} \neq \mathbf{0}$, the magnetic field is not purely transverse.

VI-6- Poynting Vector and Energy Density

From the Poynting vector and its time-averaged value:

$$\langle\vec{R}\rangle = \frac{1}{2\mu_0}\frac{k}{\omega} A^2\sin^2(\alpha y)\vec{e}_x$$

The time-averaged energy density is:

$$\langle \omega \rangle_t = \frac{1}{2}\left[\varepsilon_0 \frac{E^2}{2} + \frac{B^2}{2\mu_0}\right]$$

This simplifies to:

$$\langle \omega \rangle_t = \frac{1}{2}\left[\frac{A^2}{c^2}\sin^2(\alpha y) + \frac{A^2}{\omega^2}\alpha^2\cos^2(\alpha y) + k^2\sin^2(\alpha y))\right]$$

The spatially averaged energy density is:

$$\langle \omega \rangle_{t,y} = \frac{A^2}{4\mu_0 c^2}$$

Similarly, the spatially averaged Poynting vector is:

$$\langle \vec{R} \rangle_{t,y} = \frac{A^2}{4\mu_0 c^2}\frac{k}{\omega}\vec{e}_x$$

Thus, we obtain:

$$\langle \vec{R} \rangle_{t,y} = V_g \langle \omega \rangle_{t,y} \vec{e}_x$$

This result shows that electromagnetic energy propagates in the waveguide at the group velocity, in the direction of the wave propagation.

Exercises and solved problems

Exercice 1

On considère une onde électromagnétique sinusoïdale se propageant dans le vide est caractérisé par le champ électrique : $\vec{E} = E_0 \cos(\omega t + ky)\,\vec{e}_z$

Où k et E_0 sont supérieures à 0. La fréquence de l'onde $f = 4.10^9\ Hz.$

1- Ecrire les équations de Maxwell dans le vide en absence de charges et de courants.

2- Déduire le champ magnétique $\vec{B}$ de l'onde.

3- Déterminer la densité d'énergie électromagnétique ω_{EM} et le vecteur de Poynting $\vec{R}$.

4- Quelle est la relation entre ω_{EM} et $\vec{R}$.

Solution:

1- In a vacuum, in the absence of charges and currents, Maxwell's equations are written as:

$$\vec{\nabla}.\vec{E} = 0\ ,\vec{\nabla}.\vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}\ and\ \vec{\nabla} \times \vec{B} = \frac{1}{C^2}\frac{\partial \vec{E}}{\partial t}$$

2- In the case of a sinusoidal and uniform plane wave:

$$\vec{B} = \frac{\vec{k}\times\vec{E}}{\omega} = -\frac{kE_0}{\omega}\cos(\omega t + ky)\overrightarrow{e_y} \times \overrightarrow{e_z} = - -\frac{kE_0}{\omega}\cos(\omega t + ky)\overrightarrow{e_x}$$

3- The electromagnetic energy density:

$$\omega_{EM} = \left(\varepsilon_0 E^2 + \frac{B^2}{\mu_0}\right) = \frac{E_0^2}{\mu_0 C^2}\cos^2(\omega t + ky)$$

4- The Poynting vector:

$$\vec{R} = \frac{\vec{E}\times\vec{B}}{\mu_0} = -\frac{E_0^2}{\mu_0 C}\cos^2(\omega t + ky)\,\overrightarrow{e_y}$$

The relationship between the Poynting vector and the electromagnetic energy density is:

$$\vec{R} = -C\ \omega_{EM}\overrightarrow{e_y}$$

Exercice 2

In vacuum, in the absence of current and charge ($\rho = 0 \; et \; \vec{j} = 0$), a plane, monochromatic, travelling electromagnetic wave of amplitude H_0 is given by its magnetic excitation:

$$\vec{H} = H_0 \cos(\omega t - kx)\vec{e}_y$$

1. What is the nature of polarization and direction of propagation of this wave?

2. Determine the electric wave field associated with this wave. Deduce its polarization.

3. Calculate the Poynting vector $\vec{R}$ of this wave. Deduce its average value over time.

Exercise 3

Space is defined with an orthonormal coordinate system xoy with unit vectors $(\overrightarrow{e_x}, \overrightarrow{e_x}, \overrightarrow{e_y}, \overrightarrow{e_z})$. A plane sinusoidal wave propagating in vacuum (in the absence of charges and currents) is studied, with the electric field given by: $\vec{E} = E_x\overrightarrow{e_x} + E_y\overrightarrow{e_y}$

such that:

$$E_x = E_0 \cos(\omega t - \alpha(x + 2y + z))$$

where a is a positive constant.

1. What are the components of the wave vector $\vec{k}$ and its magnitude in terms of a?
2. Give the dispersion relation linking ω and $\vec{k}$.
3. Express E_y as a function of E_x.
4. Express the components of the magnetic field $\vec{B}$.
5. Determine the components of the Poynting vector $\vec{R}$, its magnitude, and its time-averaged value in terms of ε_0, c and E_0^2.

Solution:

1. The components of the wave vector $\vec{k}$ are: $\begin{pmatrix} a \\ 2a \\ a \end{pmatrix}$

and its magnitude is $\left|\vec{k} = \ a\sqrt{6}\right|$.

2. The dispersion relation is written as: $k = \frac{\omega}{V}$

3. Expression of E_y

We have:

$$\mathrm{div}\,\vec{\mathrm{E}} = 0 \Rightarrow \ \vec{\mathrm{k}}.\vec{\mathrm{E}} = 0$$

$$\Rightarrow \mathrm{E_y} = \ -\frac{1}{2}\mathrm{E_x}$$

4. According to the structure relation, we have: $\vec{B} = \frac{\vec{k}\times\vec{E}}{\omega}$

Thus,

$$\vec{B} = \begin{pmatrix} \frac{a}{\omega} \\ \frac{2a}{\omega} \\ \frac{a}{\omega} \end{pmatrix} \wedge \begin{pmatrix} E_x \\ -\frac{1}{2}E_x \\ 0 \end{pmatrix} = \frac{a}{\omega}\begin{pmatrix} \frac{1}{2} \\ 1 \\ -\frac{5}{2} \end{pmatrix} E_0 \cos(\omega t - \alpha(x + 2y + z))$$

5. The Poynting vector is defined as: $\vec{R} = \frac{\vec{E}\times\vec{B}}{\mu_0}$

$$\vec{R} = \begin{pmatrix} E_x \\ -\frac{1}{2}E_x \\ 0 \end{pmatrix} \wedge \begin{pmatrix} \frac{1}{2}E_x \\ E_x \\ -\frac{5}{2}E_x \end{pmatrix} = \frac{a}{\omega\mu_0}\begin{pmatrix} \frac{5}{4} \\ \frac{5}{2} \\ \frac{5}{4} \end{pmatrix} E_0^2 cos^2(\omega t - \alpha(x + 2y + z))$$

Its magnitude is:

$$\left|\vec{R}\right| == \frac{5}{4}c\varepsilon_0 E_x^2 = \frac{5}{4}c\varepsilon_0 E_0^2 cos^2(\omega t - \alpha(x + 2y + z))$$

Its time-averaged value is:

$$\langle\left|\vec{R}\right|\rangle = \frac{1}{T}\int_0^T \frac{5}{4}c\varepsilon_0 E_0^2 cos^2(\omega t - \alpha(x + 2y + z))$$

which simplifies to:

$$\langle|\vec{R}|\rangle = \frac{5}{8} c\varepsilon_0 E_0^2$$

Exercise 4

A plane electromagnetic wave with angular frequency ω\omegaω propagating in vacuum is given by the electric field:

$$\vec{E}_1 = A\ \cos(\omega t - kx)\,\vec{e}_y$$

where k>0 and A>0. The frequency of the wave is f = 6×10^{13} Hz.

1. Compute numerically the wavelength λ.

What is the relationship between ω and k?

2. a. Write Maxwell's equations in vacuum in the absence of charges and volume currents.

   b. Deduce the associated magnetic field $\vec{B}$ of the wave.

3. a. Determine the electromagnetic energy density ω_{EM} and the Poynting vector $\vec{R}$.

   b. What is the relationship between ω_{EM} and $\vec{R}$?

   c. Compute the time-averaged values $\langle\omega_{EM}\rangle$. and $\langle\vec{R}\rangle$.

   d. In what unit is $\vec{R}$ measured?

Numerical application: The measured modulus of $\langle\vec{R}\rangle$ is 53.2 (SI unit requested). Deduce the numerical value of $\langle\omega_{EM}\rangle$ and the amplitude A of wave (1).

4. In addition to wave (1), a second wave of the same amplitude, same frequency, and linear polarization along z propagates in vacuum but in the opposite direction to wave (1).

   a. Provide the expression of this second wave and the associated magnetic field.
   b. Both waves are superimposed. Compute the resulting electric and magnetic fields.
   c. What is the nature of the resulting wave?

Exercise 5

A plane sinusoidal electromagnetic wave, with the electric field polarized linearly along Oy, propagates along Ox in a linear, homogeneous, and isotropic medium, which is neutral, with a permittivity ε=2.56·ε_0 , permeability $\mu = \mu_0$, and zero conductivity.

Given: ε_0=8.85×10−12 F · m^{-1} and μ_0=4π×10−7 H · m^{-1}.

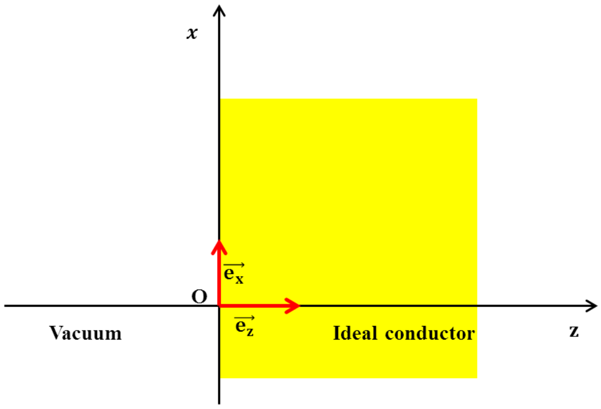
1. What is the dispersion relation $\omega = f(k)$ in this dielectric medium?
2. For a wave of angular frequency ω and wave vector$\vec{k}$, compute the associated magnetic field $\vec{B}$ given the electric field $\vec{E}$.
3. Compute, for this dielectric, the following quantities:
   - The wave speed V, the characteristic impedance Z, and its ratio $\eta$to that of vacuum Z_0.
   - The magnitude of the wave vector $\vec{k}$.
   - The wavelength λ.
   - The average energy density.

For a wave with amplitude E_0=100 kV·m^{-1} and in the following cases:

- Microwaves: f_1=1.25×10^{10} Hz.
- Visible light: f_2= 3×10^{10} Hz.

Exercise 6

A perfect conductor occupies the entire region corresponding to z>0. Its free surface, represented by the xOy plane, is in contact with air, which is considered as a vacuum.



A plane, progressive, monochromatic wave with angular frequency ω\omegaω and wave vector $\vec{k}_i = k\vec{e}_z$ is incident on the conductor. The incident wave is characterized by an electric field polarized along the x direction:

$$\vec{E}_i = E_0 e^{i(\omega t - kz)}\overrightarrow{e_x}$$

1. Fields Inside the Conductor

Under the assumption of a perfect conductor, determine the expressions for the electric field $\vec{E}$ and the magnetic field $\vec{B}$ inside the conductor.

2. Determination of Key Quantities

a. Express the reflected electric field $\vec{E}_r$ in terms of E_0 , ω, and k.

b. Determine the incident and reflected magnetic fields, $\vec{B}_i$ and $\vec{B}_r$.

c. Using continuity conditions, derive the surface charge density σ and the surface current density $\overrightarrow{J_s}$.

3. Resulting Standing Wave

a. Express the resulting electric field $\vec{E}$ and magnetic field $\vec{B}$.

b. Sketch $E(z)$ at two different time instants on the same diagram.

4. Wave Between Two Conductors

An identical conductor to the previous one is placed at $z = -L$. Show that:

a. If L is imposed, only the frequencies f_n that can be determined are possible.

b. If the wavelength λ is imposed, only the lengths L_n that can be determined are possible.

Exams with their solutions

الجمهورية الجزائرية الديمقراطية الشعبية
وزارة التعليم العالي و البحث العلمي

NATIONAL HIGHER SCHOOL FOR HYDRAULICS
"The Mujahid Abdellah ARBAOUI"

المدرسة الوطنية العليا للري
"المجاهد عبد الله عرباوي"

ⴰⵖⵉⵔⴻⵎ ⴰⵖⵏⴰⵡⴰⵏ ⵓⵍⵓⵙ ⵏ ⵜⵍⴻⵙⵓⵍⵓⵜ

Examen de Physique 4

Année universitaire : 2023-2024

Calculatrice Autorisée : NON

Exercice 1: (7 pts)

Nous considerons une corde homogene de longueur semi-infinie de masse par unité de longueur μ et tendue avec une tension T. Cette corde est terminée en $x = 0$ par une masse m qui glisse sans frottement sur une tige horizontale (Figure 1). Une onde incidente transversale sinusoidale de pulsation ω et d'amplitude U_0 se propage dans cette corde dans le sens des x croissants et se réfléchie en $x = 0$. L'ensemble du mouvement se fait dans un plan horizontal .

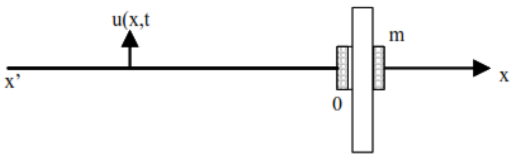


Figure 1

1- Etablir l'equation de propagation de l'onde le long d'une corde.

2- Déterminer l'expression du coefficient de réflextion en x = 0. Quel est son module R et son argument θ.

3- On recherche une solution de la forme : u(x,t) = $Ae^{j(\omega t-kx)} + B\, e^{j(\omega t+kx)}$. Que represente k ? Donner les expressions de A en fonction de U_0 et B en fonction de U_0 et θ.

4- Montrer que l'onde résultante dans la corde peut s'écrire sous la forme :

$$u\,(x,t) = U_0[1 + e^{j\varphi}]e^{j(\omega t-kx)}$$

Donner l'expression du déphasage φ en fonction de k, x et θ.

5 - Pour quelles valeurs de φ,l'amplitude de l'onde résultante est-elle maximale ?

6- Sachant que la distance séparant la masse du maximum qui lui est le plus proche est d, déterminer l'expression de d en fonction de μ, T, ω et m.

Dr. Badis RIAH

7- Dans le cas où la masse m est infinie, donner l'expression de R puis montrer qu'il s'etablit dans la corde un phénomene d'onde stationnaire. Dessiner l'allure de la corde illustarant ce phénomene d'onde stationnaire.

Exercice 2 : (6 pts)

Nous considérons un tuyau cylindrique de section (S) rempli d'un fluide de masse volumique ρ. Ce tuyau est disposé le long de l'axe x'Ox entre x = 0 et x = L. En x = 0, on place un haut parleur qui impose un mouvement sinusoidale aux particules de fluide : u (0,t))= $U_0 \sin(\omega t)$, alors qu'en x = L le tuyau est terminé par une paroi rigide (Voir figure 2).

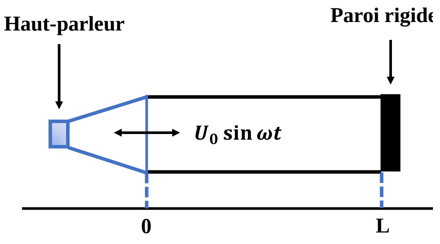


Figure 2

1- Etablir l'equation de propagation d'une onde acoustique dans un tuyau.

2- Montrer que l'onde résultante du déplacement de particules peut se mettre sous la forme suivant : u (x,t) = U (x) $\sin(\omega t)$.

3- Déduire en un point x du tuyau l'expression de la pression acoustique p(x,t).

4- Déterminer la position des points où:

- La pression est maximum P_{max}
- La pression est minimum P_{min}
- Donner les valeurs de P_{max} et P_{min}

Exercice 3 : (7 pts)

Une onde électromagéntique plane sinusoidale de pulsation ω se propage dans le vide avec une célérité c dans une direction $\vec{u}$ du plan xOy. Le champ électrique de cette onde s'ecrit en notation complexe au point M(x,y,z) à l'instant t : $\vec{E}(M) = E_0 e^{j(\omega t - az - by)} u_x$

1- Donner les équations de Maxwell dans le vide sous leurs formes locales.

2- Que représente les coefficients a et b ? Déterminer en fonction de a et b la longuer d'onde λ.

3- Etablier l'equation de propagation du champ $\vec{E}$ et déduire la relation qui lie $a, b, \omega\ et\ c$. Comment appelle-t-on cette relation ?

4- Exprimer le vecteur du champ magnétique $\vec{B}$ de l'onde étudiée. Que peut-on dire des polarisations des champs $\vec{E}$ et $\vec{B}$.

5- Calculer l'impédance caractéristique du vide z_0 définie par le rapport des amplitudes du

champ $\vec{E}$ et de l'excitation magnétique $\vec{H} = \frac{\vec{B}}{\mu_0}$, de l'onde dans le vide.

5- Déterminer les composantes du vecteur de Poynting $\vec{R}$ et sa valeur moyenne dans le temps notée $\langle R \rangle$, que représente-elle ?

Bon courage

Corrigé-type de l'examen de Physique 4

Année universitaire : 2023-2024

Calculatrice Autorisée : NON

Exercice 1 : (7 pts)

1- Etablir l'equation de propagation de l'onde le long d'une corde.

Réponse 1 : voir le chapitre 2 des ondes le long d'une corde………(1 pt)

2- Donner l'expression du coefficient de réflextion en x = 0. Quel est son module R et son argument θ.

Réponse 2 :

- *L'expression du coefficient de réflexion :* $r = \frac{\sqrt{\mu T}-im\omega}{\sqrt{\mu T}+im\omega}$ ………(0,5pt)

- *Son module et son argument :* $R = \|\vec{r}\| = 1$ *et* $\theta = -2arctg\left(\frac{m\omega}{\mu T}\right)$………(0,5 pt)

3- On recherche une solution de la forme : u(x,t) = $Ae^{j(\omega t-kx)} + B\, e^{j(\omega t+kx)}$. Que represente k ? Donner les expressions de A en fonction de U_0 et B en fonction de U_0 et θ.

Réponse 3 : k : est le vecteur d'onde, $A = U_0$ *et* $B = e^{i\theta}U_0$ ………(0,75 pt)

4- Montrer que l'onde résultante dans la corde peut s'écrire sous la forme :

u (x,t) = $U_0[1 + e^{j\varphi}]e^{j(\omega t-kx)}$ et donner l'expression du déphasage φ en fonction de k, x et θ.

Réponse 4 : $u(x,t) = U_0[1 + e^{j(\theta+2kx)}]e^{j(\omega t-kx)} \Longrightarrow \varphi = \theta + 2kx$………(1 pt)

5- Pour quelles valeurs de φ,l'amplitude de l'onde résultante est-elle maximale ?

Réponse 5 : $\varphi = n2\pi\ , n : entier$ ………(0,25 pt)

6- Sachant que la distance séparant la masse du maximum qui lui est le plus proche est d, déterminer l'expression de d en fonction de μ, T, ω et m.

Réponse 6 : $d = \frac{\sqrt{\frac{T}{\mu}}\, arctg\left(\frac{m\omega}{\mu T}\right)}{\omega}$ ………..*(1,5 pt)*

7- Dans le cas où la masse m est infinie, donner l'expression de R puis montrer qu'il s'etablit dans la corde un phénomene d'onde stationnaire. Dessiner l'allure de la corde illustarant ce phénomene d'onde stationnaire.

Réponse 7 :

si m est infinie, le coefficient R = -1 et la condition en $x = 0$ *est :* $u(0,t) = 0$ *alors*:

$\Longrightarrow u(x,t) = 2U_0 \cos kx\, e^{j\omega t}$………..*(1,5 pt)*

Exercice 2 : (6 pts)

1- Etablir l'equation de propagation de l'onde dans un tuyau.

Réponse 1 : voir le chapitre 3 des ondes acoustiques dans les fluides…….*(0,5 pt)*

2- Montrer que l'onde résultante du déplacement de particules peut se mettre sous la forme suivant : u (x,t) = U (x) $\sin(\omega t)$.

Réponse 2 : l'expression générale du déplacement des particules s'ecrit :

$$u(x,t) = U_i e^{j(\omega t - kx)} + U_r\, e^{j(\omega t + kx)} \text{……(0,25 pt)}$$

Les conditions en x=L et en x =0 s'écrivent :

- *u(L,t) =* $U_i e^{j(\omega t - kL)} + U_r\, e^{j(\omega t + kL)} = 0 \Longrightarrow U_i e^{-j(kL)} + U_r\, e^{j(kL)} = 0$ ….*(0,25 pt)*
- *u(0,t) =* $U_i e^{j(\omega t)} + U_r\, e^{j(\omega t)} = U_0 e^{j\omega t} \Longrightarrow U_i + U_r = U_0$…….*(0,25 pt)*

On déduit donc : $u(x,t) = \frac{U_0 \sin k(L-x)}{\sin kL} \sin(\omega t)$………*(1 pt)*

On reconnait ici l'expression d'un tuyau en oscilattions forcées (démonstration de cette expressione est dans le paragraphe sur les cordes vibrantes).

- *Si* $\sin k(L-x) = 0$, *on obtient des nouds de déplacement ;*….*(0,25 pt)*
- *Si* $\sin k(L-x) = \pm 1$, *on obtient un ventre de déplacements ;*………*(0, 25 pt)*
- *Si* $\sin(kL) = 0$, *on obtient une amplitude infinie (Résonance)*………*(0 ,25 pt)*

3- Déduire en un point x du tuyau l'expression de la pression acoustique p(x,t).

<u>Réponse 3 :</u> *p(x,t) s'obtient à partir de la relation : p(x,t)* = $-\frac{1}{\chi}\frac{\partial u}{\partial x}$ *soit :*

$$p(x,t) = \frac{\rho V \omega U_0 \cos[k(L-x)]}{\sin kL}\sin(\omega t)$$ *………(1 pt)*

4- Déterminer la position des points :

<u>Réponse 4 :</u>

- On obtient des maximas de pression si $\cos[k(L-x)] = \pm 1$*…….(0,5 pt)*

$\Longrightarrow x_{max} = L - n\frac{\lambda}{2}$ *et* $P_{max} = \frac{\rho V \omega U_0}{|\sin kL|}$*…….(0,5 pt) et*

- On obtient des maximas de pression si $\cos[k(L-x)] = 0$*………(0 ,5 pt)*

$\Longrightarrow x_{min} = L - (2n+1)\frac{\lambda}{4}$ *et* $P_{min} = 0$*……(0,5 pt)*

<u>Exercice 3 : (7 pts)</u>

1- Donner les équations de Maxwell dans le vide sous leurs formes locales.

<u>Réponse 1 :</u> *Les équations de Maxwell locales dans le vide sont données par :*

$\vec{\nabla}.\vec{E} = 0$ *(Théorème de Gauss)…..(0,25 pt)*

$\vec{\nabla}.\vec{B} = 0$ *(Théorème de Gauss pour le champ magnétique ou bien équation de Maxwell Thomson)……(0,25 pt)*

$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ *(Equation de Maxwell Faraday)…….(0,25 pt)*

$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j} + \varepsilon_0\, \mu_0 \frac{\partial \vec{E}}{\partial t} = \frac{1}{C^2}\frac{\partial \vec{E}}{\partial t}$ *(Théorème de Maxwell Ampère)…..(0,25 pt)*

2- Que représente les coefficients a et b ? Déterminer en fonction de a et b la longuer d'onde λ.

<u>Réponse 2 :</u> *les coefficients a et b :*

$\vec{k}.\vec{r} = ax + by = k_x + k_y$ *où a et b sont les composantes du vecteur d'onde.*

L'expression de la longueur d'onde λ : $k = \frac{2\pi}{\lambda} \Longrightarrow \lambda = \frac{2\pi}{\sqrt{a^2+b^2}}$*.........(0,5 pt)*

3- Etablier l'equation de propagation du champ $\vec{E}$ et déduire la relation qui lie a, b,ω et c. Comment appelle-t-on cette relation ?

<u>*Réponse 3 :*</u> *Les équations de propagation de* $\vec{E}$ *(voir le cours d'OEM dans le vide)*

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla}(\vec{\nabla} . \vec{E}) - \Delta\vec{E} = -\Delta\vec{E} \text{ Car } \vec{\nabla} . \vec{E} = 0 \text{ .... (1)}$$

Nous calculons le rotationnel du rotationnel, en tenant compte des équations de Maxwell :

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla} \times (-\frac{\partial \vec{B}}{\partial t}) = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B}) \text{ car } \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Selon les équations de Maxwell on 'a : $\vec{\nabla} \times \vec{B} = \frac{1}{C^2}\frac{\partial \vec{E}}{\partial t}$ cela nous donne :

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\frac{1}{C^2}\frac{\partial}{\partial t}(\frac{\partial \vec{E}}{\partial t}) = -\frac{1}{C^2}\frac{\partial^2 \vec{E}}{\partial^2 t} \text{ ...... (2)}$$

Selon (1) et (2) on trouve : $\Delta\vec{E} - \frac{1}{C^2}\frac{\partial^2 \vec{E}}{\partial^2 t} = 0$ (C'est l'équation de d'Alembert)*.....(1 pt)*

- Relation entre a, b, ω et c :

On injecte $\vec{E}$ dans l'équation de d'Alembert,

$-(a^2 + b^2)\,\vec{E} = -\frac{\omega^2}{c^2}\vec{E} \Longrightarrow (-(a^2 + b^2) + \frac{\omega^2}{c^2})\,\vec{E} = \vec{0} \Longrightarrow \sqrt{a^2 + b^2} = \frac{\omega}{c} = k$*.......(1 pt)*

C'est la relation de dispersion.

4- Exprimer le vecteur du champ magnétique $\vec{B}$ de l'onde étudiée. Que peut-on dire des polarisations des champs $\vec{E}$ et $\vec{B}$.

<u>*Réponse 4 :*</u> Expression du champ magnétique :

$\vec{B} = \frac{\vec{k} \times \vec{E}}{\omega} = \frac{E_0}{\omega} e^{j(\omega t - ax - by)}(b\vec{u}_z - a\vec{u}_y)$*........(0,5 pt)*

$\vec{E}$ est polarisé rectilignement Oz alors que $\vec{B}$ l'est dans le plan xOy, les deux champs sont orthogonaux $\vec{E}.\vec{B} = 0$*.........(0,5 pt)*

5- Calculer l'impédance caractéristique du vide z_0 définie par le rapport des amplitudes du champ $\vec{E}$ et de l'excitation magnétique $\vec{H} = \frac{\vec{B}}{\mu_0}$, de l'onde dans le vide.

Réponse 5 : Impédance du vide Z_0

$Z_0 = \frac{E_0}{H_0} = \frac{\mu_0 E_0}{B_0} = \frac{\mu_0 E_0}{E_0\frac{\sqrt{a^2+b^2}}{\omega}} = \mu_0 c$ *……….(0,5 pt)*

6- Déterminer les composantes du vecteur de Poynting $\vec{R}$ et sa valeur moyenne dans le temps notée $\langle R\rangle$, que représente-elle ?

Réponse 6 : Vecteur de Poynting $\vec{R}$

$\vec{R} = \frac{\vec{E}\times\vec{B}}{\mu_0} = \frac{E^2}{\mu_0\omega}(a\vec{u}_z - b\vec{u}_y)$ *………(0,5 pt)*

La valeur moyenne du vecteur de Poynting :

Le module de $\vec{R}$ est : $\frac{E^2}{\mu_0\omega}\sqrt{a^2+b^2} = \frac{E^2}{\mu_0 c}$ *………(0,5)*

$\langle R\rangle = \frac{1}{T}\int_0^T R\,dt = \frac{E_0^2}{\mu_0 cT}\int_0^T cos^2(\omega t - ax - by)dt \Longrightarrow \langle R\rangle = \frac{E_0^2}{2\mu_0 c}$ *………(0,5 pt)*

La valeur du vecteur de Poynting correspond à la puissance par unité de surface (intensité)*……(0 ,5 pt)*

الجمهورية الجزائرية الديمقراطية الشعبية
وزارة التعليم العالي و البحث العلمي
NATIONAL HIGHER SCHOOL FOR HYDRAULICS
"The Mujahid Abdellah ARBAOUI"
المدرسة الوطنية العليا للري
"المجاهد عبد الله عرباوي"
ⴰⵖⴻⵔⴱⴰⵣ ⴰⵖⴻⵍⵏⴰⵡ ⵉⵎⵉⵏⴻⵣ ⵏ ⵜⵎⴻⵙⵡⵉⵜ

Examen de remplacement de Physique 4

Année universitaire : 2023-2024

Calculatrice Autorisée : OUI

Exercice 1 : (7 pts) Propagation d'une onde mécanique le long d'une corde

Une corde de longueur L, de masse linéque µ, sous une tension T, est parcourue par deux ondes :$y_1(x,t) = A\cos(\omega t - kx + \varphi_1)$ et $y_2(x,t) = A\cos(\omega t + kx + \varphi_2)$

Où y est le déplacement transversal, ω la pulsation et k le module du vecteur d'onde.

1- Montrer que la corde est le siège d'onde stationnaire.

2- Montrer qu'il existe des points sur la corde qui restent toujours immobiles (nœuds).

3- Exprimer la distance Δx entre deux nœuds succsessifs en fonction de la longueur d'onde.

4- La corde étant fixée aux extrémités, montrer que le module du vecteur d'onde ne peut prendre que certaines valeurs particulieres k_n. En déduire les valeurs permises f_n de la fréquence.

5- Sur le violon, la note mi de fréquence fondamentale $f = 660\ Hz$ est obtenue en faisant vibrer une corde de longueur L = 33 cm et de masse linéique µ = 5,46 .$10^{-1} kg/m$. Dans ce cas, calculer la tension de la corde.

On donne les transformations trigonométrique :

$$\cos a + \cos b = 2\cos\left(\frac{b+a}{2}\right)\cos\left(\frac{b-a}{2}\right) \text{ et } \cos(a+b) = \cos a\cos b - \sin a \sin b$$

Exercice 2 : (13 pts)

Partie A : (6 pts) Propagation d'une onde électromagéntique dans le vide

Une onde électromagnétique plane sinusoidale et linéairement polarisée, correspondant à la lumiere de longueur d'onde λ = 5 $10^{-7}m$ se propage dans le vide. Son intensité (puissance par unité de surface) est I = 0,1 W.m^{-2}. Sa direction de propagation se trouve dans le plan xOy et fait un angle $\alpha = 30°$ avec l'axe Ox. La direction de polarisation du champ électrique est suivant Oz.

1- Ecrire les expressions littérales et numériques du champ électrique, du champ magéntique et du vecteur de Poynting. Faire un dessin représentant les champs électriques et magéntiques et le vecteur de cette onde. On donne $I = \frac{\langle |E^2| \rangle}{Z_0}$.

2- Calculer la puissance moyenne reçue par un cadre carré de 1 cm de coté perpendiculaire à la direction de propagation.

On donne la vitesse de propagation de la lumiere dans le vide c = 3 10^8 m/s ainsi que l'impédance caractéristique du vide Z_0 = 377 Ω.

Partie B : (7 pts) Réflexion d'une onde électromagnétique dans un conducteur parfait

Une onde plane électromagéntique incidente se propage dans le vide (milieu 1) et rencontre un conducteur parfait (milieu 2) de conductivité γ qui occupe le demi-espace (z>0).

L'onde incidente est décrite par le champ électrique :

$$\vec{E_i} = E_0 \cos(\omega t - k.z)\vec{u}_x - E_0 \sin(\omega t - k.z)\vec{u}_y$$

1- Donner les champs électrique $\vec{E}_2$ et magéntique $\vec{B}_2$ dans le conducteur parfait. Justifier.

2- Trouver l'expression du champ électrique réfléchie $\vec{E}_r$. Calculer le champ magéntique $\vec{B}_r$ associé.

3- Déduire le champ $(\vec{E}_1, \vec{B}_2)$ résultant de la superposition des ondes incidente et réfléchie.

4- L'onde résultante correpond-elle à une onde progressive ? Justifier.

5- Déterminer le vecteur de Poynting $\vec{R}$ correspondant à cette onde. Commenter.

Bon courage

الجمهورية الجزائرية الديمقراطية الشعبية

وزارة التعليم العالي و البحث العلمي

NATIONAL HIGHER SCHOOL FOR HYDRAULICS
"The Mujahid Abdellah ARBAOUI"

المدرسة الوطنية العليا للري
"المجاهد عبد الله عرباوي"
ⴰⵖⵓⵔⵟⴰⵥ ⴰⵖⵓⵏⵍⴰⵡ ⵉⵎⵉⵍⵓⵅ ⵍ ⵜⵎⵓⴹⵡⵉⵜ

Corrigé-type de l'examen de remplacement de Physique 4

Année universitaire : 2023-2024

Exercice 1 : (7 pts)

1- Montrer que la corde est le siège d'onde stationnaire.

Réponse 1 :

$$y(x,t) = y_1(x,t) + y_2(x,t) \Longrightarrow A[\cos(\omega t - kx + \varphi_1) + \cos(\omega t + kx + \varphi_2)]$$

$$y(x,t) = 2A\cos\left(\omega t + \frac{\varphi_1 + \varphi_2}{2}\right)\cos\left(kx + \frac{\varphi_2 - \varphi_1}{2}\right)$$

$y(x,t) = 2A\cos(kx + \psi)\cos(\omega t + \varphi)$ *représente une onde staionnaire…......(1 pts)*

2- Montrer qu'il existe des points sur la corde qui restent toujours immobiles (nœuds).

Réponse 2 :

Les nœuds sont les points tels que $y(x,t) = 0\ \forall\ t$*, c'est-à-dire que :* $\cos(kx + \psi) = 0$

D'où : $kx + \psi = \frac{(2n+1)\pi}{2} \Longrightarrow x_n = \left[\frac{(2n+1)\pi}{2} - \psi\right]/k$*………......(1 pts)*

3- Exprimer la distance Δx entre deux nœuds succsessifs en fonction de la longueur d'onde.

Réponse 3 :

$\Delta x = x_{n+1} + x_n = \frac{[2(n+1)+1]\pi}{2k} - \frac{(2n+1)\pi}{2k} = \frac{\pi}{k}; k = \frac{2\pi}{\lambda} \Longrightarrow \Delta x = \frac{\lambda}{2}$*……….(1 pts)*

Réponse 4 :

La corde étant fixée à ses extrémités $x = 0$ et $x = L$, l'onde stationnaire a un nœud en x = 0 et un nœud en $x = L$.

Les conditions aux limites :

- $y(0,t) = 0 \Longrightarrow 2A\cos\psi\cos(\omega t + \varphi) = 0 \Longrightarrow \cos\psi = 0 \Longrightarrow \psi = \frac{(2n+1)\pi}{2}$ *(0,5 pts)*

- $y(L,t) = 0$

$\Longrightarrow 2\,A\cos\left[kL + \frac{(2n+1)\pi}{2}\right]\cos(\omega t + \varphi) = -2A\sin kL\sin\left[\frac{(2n+1)\pi}{2}\right]\cos(\omega t + \varphi)$ *(0,5 pt)*

$\Longrightarrow \sin kL = 0 \Longrightarrow kL = n\pi \Longrightarrow k_n = \frac{n\pi}{L} = \frac{\omega_n}{V} = \frac{2\pi f_n}{V}$……*(1 pt)*

D'où : $f_n = n\frac{V}{2L}$ où V est la vitesse de phase de l'onde ……*(1 pt)*

Réponse 5 :

Corde du violon : L = 33 cm, μ = $5{,}4610^{-4}kg/m$

La fréquence fondamental est : $f = \frac{V}{2L} = \frac{1}{2L}\sqrt{\frac{T}{\mu}} = 660\ Hz$ ……*(0,5 pt)*

D'où : $T = 4L^2\mu f^2$ ……*(0,25 pt)* A.N : T = 104 N ……*(0,25 pt)*

Exercice 2 : (13 pts)

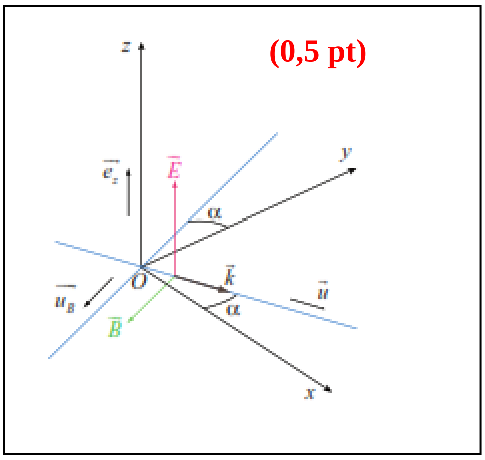
Partie A :

1- Le vecteur d'onde s'ecrit sous la forme :

$\vec{k} = k\,(\cos\alpha\vec{e}_x + \sin\alpha\vec{e}_y) = k\frac{\sqrt{3}}{2}\vec{e}_x + \frac{k}{2}\vec{e}_y = k\vec{u}$ (0,5 pt)

Avec $k = \frac{2\pi}{\lambda} = 1{,}25.10^7 m^{-1}$… (0,5 pt) . Donc

$$\vec{k} = \begin{cases} k_x = 1{,}09.10^7 m^{-1} \\ k_y = 6{,}30.10^6 m^{-1} \\ k_z = 0 \end{cases}$$ ……..(0,5 pt)



Le champ électrique étant polarisé linéairement suivant O :

On'a :

$\vec{E} = \vec{E}_0 e^{j(\omega t - kx\cos\alpha - ky\sin\alpha)} = E_0 e^{j(\omega t - kx\cos\alpha - ky\sin\alpha)}\vec{e_z}$………(0,5 pt)

On a: $I = \frac{\langle\vec{E}^2\rangle}{Z_0} = \frac{1}{2}\frac{E_0^2}{Z_0}$. On déduit l'amplitude E_0: $E_0 = \sqrt{2Z_0 I} = 8{,}68\ Vm^{-1}$……(0,5 pt)

D'où : $\vec{E} = 8{,}68\, e^{j(\omega t - k\frac{\sqrt{3}}{2}x - \frac{k}{2}y)}\overrightarrow{e_z}\ (Vm^{-1})$.....(0,5 pt)

Le champ magéntique s'ecrit :

$\vec{B} = \frac{\vec{k}\times\vec{E}}{\omega} = \frac{E_0}{c}(\sin\alpha\overrightarrow{e_x} - \cos\alpha\,\overrightarrow{e_y})\, e^{j(\omega t - kx\cos\alpha - ky\sin\alpha)}$.....(0,25 pt)

Où encore :

$$\vec{B} = 2{,}89.10^{-8}(\frac{1}{2}\overrightarrow{e_x} - \frac{\sqrt{3}}{2}\overrightarrow{e_y})\, e^{j(\omega t - k\frac{\sqrt{3}}{2}x - \frac{k}{2}y)}\ (T)$$ .....(0,25 pt)

$$\vec{B} = 2{,}89.10^{-8}\, e^{j(\omega t - k\frac{\sqrt{3}}{2}x - \frac{k}{2}y)}\overrightarrow{u_B}$$.....(0,25 pt)

Avec $\overrightarrow{u_B} = \frac{1}{2}\overrightarrow{e_x} - \frac{\sqrt{3}}{2}\overrightarrow{e_y}$. Le vecteur de Poynting s'écrit :

$\vec{R} = \frac{\vec{E}\times\vec{B}}{\mu_0} = (\cos\alpha\overrightarrow{e_x} - \sin\alpha\,\overrightarrow{e_y})\frac{E_0^2}{\mu_0 c}\, cos^2(\omega t - kx\cos\alpha - ky\sin\alpha)$.....(0,25 pt)

$= \frac{E_0^2}{\mu_0 c}\, cos^2(\omega t - kx\cos\alpha - ky\sin\alpha)\ \vec{u}$.....(0,25 pt)

$= \frac{E_0^2}{Z_0}\, cos^2(\omega t - kx\cos\alpha - ky\sin\alpha)\ \vec{u}$.....(0,25 pt)

Ou encore : $\vec{R} = 0{,}2\, cos^2(\omega t - k\frac{\sqrt{3}}{2}x - \frac{k}{2}y)\,\vec{u}\ \ (W.m^{-2})$ .....(0,5 pt)

Où $\vec{u} = \cos\alpha\overrightarrow{e_x} - \sin\alpha\,\overrightarrow{e_y}$ est un vecteur unitaire suivant la direction de propagation

2- La puissance à travers un cadre carré de section S = a^2, perpendiculaire à $\vec{u}$ est :

$P = \iint_S \vec{R}\, d\vec{S} = |\vec{R}|S$.....(0,25 pt)

Donc : $\langle P\rangle = \langle\vec{R}\,\rangle\, S = I * a^2 = 10^{-5}W$.....(0,25 pt)

Partie B :

1- Un conducteur parfait est un conducteur dont la conductivité γ est infinie

$\Longrightarrow$ Le champ électrique est nul $\overrightarrow{E_2} = 0$..........(0,5 pt)

Sinon la puissance par unité de volume dissipée par effet Joule $P_{joule} = \vec{J}.\vec{E} = \gamma\,\overrightarrow{E^2}$ serait infinie, ce qui est absurde, ...(0,5 pt)

L'equation de Maxwell – Fradaya $-\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times \vec{E}$ donne $\vec{B}$ statique $\Longrightarrow \vec{B_2} = 0$.......(0,5 pt)

2- D'après les conditions aux limites :

$\vec{E_1}$(z = 0) = $\vec{E_2}$(z = 0) = $\vec{0}$..........(0,5 pt)

$\vec{E_1}$(z = 0) = $\vec{E_\iota}$(z=0) + $\vec{E_r}$(z = 0)= $\vec{0}$..........(0,5 pt)

$\overrightarrow{E_{rx}}$(z = 0) = - $\overrightarrow{E_{\iota y}}$(z = 0) ....(0,5 pt)

$\overrightarrow{E_{ry}}$(z = 0) = - $\overrightarrow{E_{\iota y}}$(z = 0) ....(0,5 pt)

D'où : $\vec{E_r} = -E_0 \cos(\omega t + k.z)\vec{u_x} + E_0 \sin(\omega t + k.z)\vec{u_y}$ ....(0,5 pt)

$\vec{B_r} = \frac{E_0}{c} \sin(\omega t + k.z)\vec{u_x} + \frac{E_0}{c} \cos(\omega t + k.z)\vec{u_y}$ ....(0,5 pt)

3- Le champ résultant :

$\vec{E_1} = \vec{E_\iota} + \vec{E_r} = 2E_0 \sin(kz)(\sin(\omega t)\vec{u_x} + \cos(\omega t)\vec{u_y})$ ....(0,5 pt)

$\vec{B_2} = \vec{B_\iota} + \vec{B_r} = \frac{2E_0}{c} \cos(kz)(\sin(\omega t)\vec{u_x} + \cos(\omega t)\vec{u_y})$ ....(0,5 pt)

4- Le champ réulsant ne correspond pas à la propagation d'une onde car les variables spatiale et temporelle sont séparées. Le critere de propagation n'est par vérifié, l'onde résultante n'est par progressive mais stationnaire. ....(0,5 pt)

5- Vecteur de Poynting : $\vec{R} = \frac{\vec{E_1} \times \vec{B_2}}{c} = \vec{0}$ ....(0,5 pt)

Le vecteur de Poynting est partout égal au vecteur nul, ce qui signifie qu'il n'y a pas un transport de l'énergie : l'énergie transportée par l'onde incidente égale à celle transportée par l'onde réfléchie dans le sens inverse...........(0,5 pt).

الجمهورية الجزائرية الديمقراطية الشعبية
وزارة التعليم العالي و البحث العلمي

NATIONAL HIGHER SCHOOL FOR HYDRAULICS "The Mujahid Abdellah ARBAOUI"

المدرسة الوطنية العليا للري "المجاهد عبد الله عرباوي"

ⴰⵖⵉⵔⵎⴰⵏ ⴰⵖⵏⵎⴰⵡ ⵉⵏⵉⵍⵉⵏ ⵏ ⵜⴰⵡⵔⵉⵜ

Examen de rattrapage de Physique 4

Durée : 02H00 **Coefficient : 4**

Calculatrice Autorisée : NON

Exercice 1 : (6 pts)

Sur la figure ci-dessous, une corde de longueur L est tendue en deux points situés respectivement en x = 0 et x =L. Le point situé en x =0 est fixe et le point situé en x = L est relié à un ressort de constante de raideur K. La tension de la corde T. La corde est horizontale à l'équilibre et on pourra négliger son poids. On étudie les ondes transverses stationnaires sinusoïdale de pulsation ω. La vitesse de propagation V.

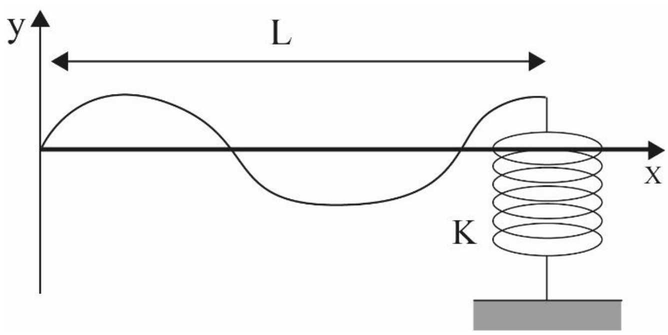


Figure 1

1) Ecrire le déplacement y(x,t) en un point quelconque d'abscisse x et à l'instant t.

2) En écrivant la condition limite pour x = 0, montrer que y (x,t) peut se mettre sous la forme : y(x,t) = $f(x)e^{j\omega t}$, où $f(x)$ est une fonction que l'on explicitera.

3) Ecrire les conditions aux limites en x =L.

4) Montrer que les pulsations propres doivent vérifier la condition : tan $(\frac{\omega L}{V}) = C\ \frac{\omega L}{V}$ où C est une constante que l'on précisera.

5) Déterminer les trois premières pulsations propres $\omega_1, \omega_2 et\ \omega_3$ dans le cas où $\frac{T}{KL} \longrightarrow \infty$.

Dr. Badis RIAH

Exercice 2 : (7 pts)

Soit un tuyau de longueur L et de section S rempli d'un fluide de masse volumique ρ_0. Un piston, placé en x = 0, vibre selon la loi : $p(0,t) = P_0 e^{j\omega t}$ (voir Figure 2).

1) Donner l'expression de l'onde résultante de la pression acoustique en un pont x du tuyau.

2) On rappelle que le coefficient de réflexion de la pression acoustique est donné par : $r_p = \frac{Z_L - Z_c}{Z_L - Z_c}$,où Z_C et Z_L représentent respectivement l'impédance caractéristique du fluide et celle de la paroi rigide. Ecrire les conditions aux limites en x = 0 et x = L.

3) Montrer que la pression acoustique peut s'exprimer p(x,t) = f(x) $e^{j\omega t}$. Spécifier l'expression de f(x).

4) En déduire les positions des maxima (ventres) et des minima (nœuds) de la pression acoustique. Que peut-on dire de la pression acoustique sur la paroi rigide ?

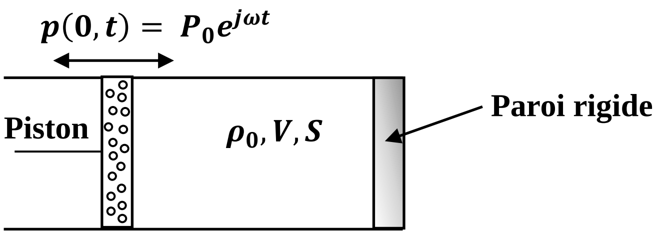


Figure 2

Exercice 3 : (7 pts)

On considere dans le vide, dans l'espace rapporté à un repere orthonormé Oxyz, muni des vecteurs de base $(\vec{e}_x, \vec{e}_y, \vec{e}_z)$. Deux ondes électromagnétiques dont les champs électriques sont donnés par :

$$\vec{E}_1(x,t) = E_0 e^{j(\omega t - kx)} \vec{e}_z \quad , \qquad \vec{E}_2(x,t) = E_0 e^{j(\omega t - ky)} \vec{e}_z$$

On demmande :

1- Les vecteurs d'onde $\vec{k}_1$ et $\vec{k}_2$ des ondes 1 et 2.

2- La direction de propagation ainsi que la polarisation (longitudinale ou transversale).

3- Les champs magnétiques $\vec{B}_1(x,t)$ et $\vec{B}_2(x,t)$ correspondants.

4- Donner les expressions de $\vec{E}_1(x,t)\ et\ \vec{E}_2(x,t)$ en notation réelle puis déduire le champ électrique résultant $\vec{E} = \vec{E}_1 + \vec{E}_2$ en notation réelle et le mettre sous la forme :

$\vec{E}(x,t) = \vec{\Sigma}(x,y)\cos[\omega t - \varphi(x,y)]$

a- Donner les expressions de $\vec{\Sigma}(x,y)$ et $\varphi(x,y)$

b- Quel est le vecteur d'onde $\vec{k}'$ de cette onde

c- Dans quelle direction de l'espace observe-t-on un phénomene d'ondes stationnaires pour $\vec{E}$? Quels sont alors les lieux géométriques des noueds et des ventres.

On donne : $\mathbf{\cos(a) + \cos(b) = 2\cos\frac{a+b}{2}\cos\frac{a-b}{2}}$

Bon courage

NATIONAL HIGHER SCHOOL FOR HYDRAULICS
"The Mujahid Abdellah ARBAOUI"

المدرسة الوطنية العليا للري
"المجاهد عبد الله عرباوي"

ⴰⵖⵔⴰⵙⴰⵅ ⴰⵖⵏⴰⵎⴰⵏ ⵏ ⵜⵎⵙⵔⴰⵏⵜ

Corrigé-type de l'examen de rattrapage de Physique 4

Durée : 02H00 **Coefficient : 4**

Calculatrice Autorisée : NON

Exercice 1 : (6 pts)

1) $y(x,t) = u_i e^{j(\omega t-kx)} + u_r e^{j(\omega t+kx)}$..(0,5 pt)

2) En x = 0 : $y(0,t) = u_i e^{j\omega t} + u_r e^{j\omega t} = 0 \Longrightarrow u_r = u_i = -u_0$........................(0.5 pt)

$y(x,t) = u_0 e^{j(\omega t-kx)} - u_0 e^{j(\omega t+kx)} = u_0 e^{j\omega t}(e^{-jkx} - e^{jkx}) = u_0 e^{j\omega t}(-2j\sin kx)$

$y(x,t) = -2ju_0 \sin(kx)\, e^{j\omega t} = f(x)g(t)$..(1 pt)

3) En x =L : on applique la relation fondamentale de la dynamique : ....(0.5 pt)

F de la corde + F du ressort = 0 $\Longrightarrow -T \left.\frac{\partial y}{\partial x}\right|_{x=L} - C\, y(x,t) = 0 \Longrightarrow -T \left.\frac{\partial y}{\partial x}\right|_{x=L} = C\, y(x,t)$

Cela donne : - T k cos (kL) = k sin (kL) $\Longrightarrow$ tg (kL) = $-Tk/C$ (C est la raideur K du ressort).....(0,5 pt)

4) k $=\frac{\omega}{V} \Longrightarrow tg\left(\frac{\omega L}{V}\right) = -\frac{T}{CL}\left(\frac{\omega L}{V}\right)$

tg(x) = A x , A = $\frac{-T}{CL}$, x = $\frac{\omega L}{V}$...(0.5 pt)

5) $\frac{T}{CL} \to \infty \Longrightarrow$ C ou la raideur K $\to \infty$ par de ressort $\to$ extremité x = L libre.......(0.5 pt)

a) $x_1 = \frac{\pi}{2} \Longrightarrow \frac{L\omega_1}{V} = \frac{\pi}{2} \Longrightarrow \omega_1 = \frac{\pi V}{2L} \Longrightarrow \lambda = 4L$ ....(1pt)

b) $x_2 = \frac{3\pi}{2} \Longrightarrow \frac{L\omega_2}{V} = \frac{3\pi}{2} \Longrightarrow \omega_2 = \frac{3\pi V}{2L} \Longrightarrow \lambda = \frac{4}{3}L$ ....(1pt)

Exercice 2 : (7 pts)

1) Expression de l'onde résultante

$p(x,t) = A\, e^{j(\omega t - kx)} + B\, e^{j(\omega t + kx)}$ ....(0.5 pt)

2) Conditions aux limites

$x = 0 : p(0,t) = A + B = P_0$ ....(0.5 pt)

$x = L : p(L,t) = A\, e^{j(\omega t - kL)} + B\, e^{j(\omega t + kL)} = 0$ ....(0.5 pt)

3) Pression acoustique

$r_p = \frac{Z_L - Z_C}{Z_L + Z_C}$ pour une paroi rigide $Z_L = \infty \Longrightarrow r_p = 1$ ....(0.5 pt)

D'où $\frac{B}{A} e^{2jkL} = 1$ ....(0.5 pt)

$A = \frac{P_0}{1 + e^{-2jkL}}$ ....(0.5 pt)

$B = \frac{P_0}{1 + e^{-2jkL}}\, e^{2jkL}$ ....(0.5 pt)

$$P(x,t) = \frac{P_0}{1 + e^{-2jkL}}\, e^{j(\omega t - kx)} + \frac{P_0}{1 + e^{-2jkL}}\, e^{-2jkL}\, e^{j(\omega t + kx)}$$

$$p(x,t) = \frac{P_0 e^{j\omega t}}{e^{jkL} + e^{-jkL}} [e^{-jkx} e^{jkL} + e^{jkx} e^{-jkL}]$$

$$P(x,t) = P_0 e^{j\omega t} [\frac{e^{jk(L-x)} + e^{-jk(L-x)}}{e^{jkL} + e^{-jkL}}]$$

$p(x,t) = P_0 e^{j\omega t} \frac{\cos[k(L-x)]}{\cos(kL)}$ ....(1.5pt)

Avec $f(x) = P_0 \frac{\cos[k(L-x)]}{\cos(kL)}$ ....(0.5 pt)

4) Positions

Maxima

$\cos[k(L-x)] = \pm 1 \Longrightarrow x_n = L - \frac{n\lambda}{2}$ ....(0.5 pt)

Minima

$\cos[k(L-x)] = 0 \Longrightarrow x_n = L - \frac{(2n+1)\lambda}{4}$ ....(0.5 pt)

En x = L, la pression acoustique est maximale ....(0.5 pt)

Exercice 3 : (6 pts)

1- $\vec{k}_1 = k.\vec{e}_x$ $(0.5\,pt)$ et $\vec{k}_2 = k.\vec{e}_y$ $(0.5\,pt)$ $avec\ k = \omega/c$

2- $\vec{E}_1$: le champ est polarisé suivant z et la direction de propagation est suivant x donc l'onde est transeversal. $(0.5\,pt)$

$-\vec{E}_2$: le champ est polarisé suivant z et la direction de propagation est suivant y donc l'onde est transeversal$(0.5\,pt)$

3- $\vec{B}_1(x,t) = \frac{E_0}{c} e^{j(\omega t - kx)}\, \vec{e}_y$ $(0.5\,pt)$ et $\vec{B}_2(x,t) = \frac{E_0}{c} e^{j(\omega t - ky)}\, \vec{e}_x$ $(0.5\,pt)$

4- $\vec{E}(x,t) = E_0[\cos(\omega t - kx) + \cos(\omega t - ky)]\,\vec{e}_z$

$$\Rightarrow \vec{E}(x,t) = 2E_0\left[\cos\left(k\frac{x-y}{2}\right)\cos\left(\omega t - k\frac{x+y}{2}\right)\right]\vec{e}_z \ (0.5\,pt)$$

a- $\vec{\Sigma}(x,y) = 2E_0\left[\cos\left(k\frac{x-y}{2}\right)\right]\vec{e}_z$ et $\varphi\,(x,y) = k\frac{x+y}{2}$ $(0.5\,pt)$

b- $\vec{k}' = \frac{k}{2}\,\vec{e}_x + \frac{k}{2}\,\vec{e}_y$ $(0.5\,pt)$

c- On observe un phénomene d'ondes stationnaires suivant la droite (x-y) = cst $(0.5\,pt)$

- Avec des nœuds pour : k $\frac{x-y}{2} = (2n+1)\frac{\pi}{2}$ $(0.5\,pt)$
- Avec des ventres pour : k $\frac{x-y}{2} = n\pi$ $(0.5\,pt)$

الجمهورية الجزائرية الديمقراطية الشعبية

وزارة التعليم العالي و البحث العلمي

NATIONAL HIGHER SCHOOL FOR HYDRAULICS
"The MujahidAbdellah ARBAOUI"

المدرسة الوطنية العليا للري
"المجاهد عبد الله عرباوي"

Concours National d'accès au second cycle des écoles supérieures

Année universitaire 2023/2024

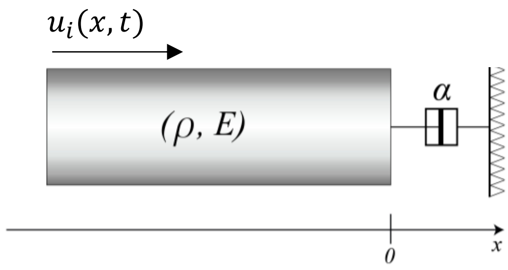
SUJET

Domaine : ST **Matière : Physique**

Durée : 01H15 **Coefficient : 1**

Calculatrice Autorisée : OUI

<u>Exercice 1 :</u> (4 points)



Consider a semi-infinite solid bar of density ρ and coefficient of rigidity E, whose end $x = 0$ is connected to a damper with. An incident wave coming from the left gives rise to a reflected wave at $x = 0$. we assume $Z_c \geq \alpha$, where Z_c specify the characteristic impedance of the solid rod.

4- a- Provide the reflection coefficient r for displacement at $x = 0$ as function of α *and* Z_c.

b- Montrer que l'onde résultante de déplacement se mettre sous la forme :

$$u(x,t) = U(x)e^{j(\omega t - kx)}$$

Où $U(x)$ est une fonction complexe à déterminer en fonction de r et des données du problème ;

c- Déterminer les positions et l'amplitude U_{min} des minimas d'amplitude (nœuds) ainsi que les positions et l'amplitude des maximas d'amplitude (ventres). En déduire le taux d'ondes stationnaires T.O.S défini par $\tau = \frac{U_{max}}{U_{min}}$.

Dr. Badis RIAH

5- On suppose maintenant que maintenant que $\alpha = \infty$.

   c- Que devient l'expression de $u(x,t)$?

   d- Déterminer les points qui vibrent avec une amplitude maximale (ventres) et minimale (nœuds).

6- Quelle est la nature de $u(x,t)$ dans le cas où $\alpha = Z_c$?

<u>Exercice 2 :</u> (3 points)

On considère une onde électromagnétique sinusoïdale se propageant dans le vide est caractérisé par le champ électrique : $\vec{E} = E_0 \cos(\omega t + ky)\, \vec{e}_z$

Où k et E_0 sont supérieures à 0. La fréquence de l'onde $f = 4.10^9\ Hz$.

6- Ecrire les équations de Maxwell dans le vide en absence de charges et de courants.

7- Déduire le champ magnétique $\vec{B}$ de l'onde.

8- Déterminer la densité d'énergie électromagnétique ω_{EM} et le vecteur de Poynting $\vec{R}$.

9- Quelle est la relation entre ω_{EM} et $\vec{R}$.

الجمهورية الجزائرية الديمقراطية الشعبية

وزارة التعليم العالي و البحث العلمي

NATIONAL HIGHER SCHOOL FOR HYDRAULICS "The Mujahid Abdellah ARBAOUI"

المدرسة الوطنية العليا للري "المجاهد عبد الله عرباوي"

ⵜⵉⵏⵎⵍ ⵜⴰⵏⴰⵎⵓⵔⵜ ⵜⴰⵏⴼⵍⴰⵏⵜ

Concours National d'accès au second cycle des écoles supérieures

Année universitaire 2023/2024

Corrigé type

Domaine :ST **Matière : Physique**

Durée : 1h15min **Coefficient : 1**

<u>Exercice 1:</u> (4 points)

Solution:

1-a. The displacement reflection coefficient is expressed as: r = $\frac{z_c-\alpha}{z_c+\alpha}$ (0.25 pt)

b. The resulting wave in terms of displacement is given by:

$$\boldsymbol{y}(\mathrm{x},\mathrm{t}) = y_0\left[e^{j(\omega t-kx)} + re^{j(\omega t-kx)}\right] = y_0(1+re^{2jkx})e^{j(\omega t-kx)}$$ (0.5 pt)

Where $Y(x) = y_0(1+re^{2jkx})$ (0.25 pt)

c. The magnitude of the displacement amplitude is given by:

$|Y(x)| = y_0|1+re^{2jkx}| = y_0\sqrt{1+r^2+2r\cos(2kx)}$ (0.25 pt)

- The amplitude reaches its maxima when $\cos 2kx = +1$ so:

  $|Y(x)| = Y_{max} = 1+r$ (0.25 pt)

Let : $\cos 2kx = +1 \Longrightarrow 2kx_{max} = 2n\pi \Longrightarrow x_{max} = n\frac{\lambda}{2}$ $(n = 0,-1,-2\ \ldots.)$ (0.25 pt)

- The amplitude reaches its minima whencos $2kx = -1$ so:

  $|Y(x)| = Y_{min} = 1-r$ (0.25 pt)

Let: $\cos 2kx = -1 \Longrightarrow x_{min} = (2n-1)\frac{\lambda}{4}(n = 0, -1, -2 \ldots..)$ (0.25 pt)

From this, the Standing Wave Ratio (SWR) is deduced: $\tau = \frac{Y_{max}}{Y_{min}}$ (0.25 pt)

2- a. If $\alpha = 0, r = +1$

$$y(x,t) = y_0\left[e^{j(\omega t - kx)} + e^{j(\omega t + kx)}\right] = 2y_0 \cos(kx) e^{j\omega t} \text{ (0.25 pt)}$$

b. The antinodes are located at positions where $\cos(kx) = \pm 1$, let $kx_{max} = n\pi$. (0.25 pt)

Where: $x_{max} = \frac{n\lambda}{2}$ $(n = 0, -1, -2, \ldots)$ (0.25 pt)

The nodes occur at positions where $\cos(kx) = \pm 0$, let $kx_{min} = \frac{(2n-1)\pi}{2}$ (0.25 pt)

Where: $x_{min} = \frac{(2n-1)\lambda}{4}$ $(n = 0, -1, -2, \ldots)$ (0.25 pt)

3- if $\alpha = Z_c, r = 0$:

$y(x,t) = y_0 e^{j(\omega t - kx)}$ (0.25 pt)

Exercice 2 (3 points)

1- Dans le vide, en absence de charges et de courants, les équations de Maxwell s'écrivent :

$$\vec{\nabla}.\vec{E} = 0 \text{ (0.25 pt) }, \vec{\nabla}.\vec{B} = 0 \text{ (0.25 pt)}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \text{ (0.25 pt) } \; et \; \vec{\nabla} \times \vec{B} = \frac{1}{C^2}\frac{\partial \vec{E}}{\partial t} \text{ (0.25 pt)}$$

2-Dans le cas d'une onde plane sinusoïdale et uniforme :

$$\vec{B} = \frac{\vec{k} \times \vec{E}}{\omega} = -\frac{kE_0}{\omega}\cos(\omega t + ky)\overrightarrow{e_y} \times \overrightarrow{e_z} = - -\frac{kE_0}{\omega}\cos(\omega t + ky)\overrightarrow{e_x} \text{ (0.5 pt)}$$

3- La densité d'énergie électromagnétique :

$$\omega_{EM} = \left(\varepsilon_0 E^2 + \frac{B^2}{\mu_0}\right) = \frac{E_0^2}{\mu_0 C^2}\cos^2(\omega t + ky) \text{ (0.5 pt)}$$

4- Le vecteur de Poynting :

$$\vec{R} = \frac{\vec{E} \times \vec{B}}{\mu_0} = -\frac{E_0^2}{\mu_0 C}\cos^2(\omega t + ky)\,\overrightarrow{e_y} \text{ (0.5 pt)}$$

La relation entre le vecteur de Poynting et la densité d'énergie électromagnétique est :

$$\vec{R} = -C\,\omega_{EM}\overrightarrow{e_y} \text{ (0.5 pt)}$$

References

References:

[1] B.Demri, vibrations et ondes, Ecole Nationale Supérieure de Polytechnique d'Alger, 2021.

[2] N. Maghlaoui, vibrations et ondes, *resumé de cours et exercices et problemes corigés,studpcu, 2023.*

[4] Gorge C. King, *Vibrations and Waves*, Wiley Inc, ISBN: 978-1-2118-68178-7, 2009.

[5] Tamer Becherrawy, *Mechanical and Electromagnetic Vibrations and Waves*, Wiley Inc, ISTE Ltd, ISBN: 978-1-84821-283-1, 2012.

[6] E. Amzallag, J. Cipriani, N. Piccioli, *Rappel de Cours et Exercices Corrigés de Physique, Tome 4, Ondes*, Edisciences International, Paris, 1997.

[7] R. Gabillard, *Vibrations et Phénomènes de Propagation*, Dunod, Paris, 1969.

[8] H. Djelouah, *Vibrations et Ondes : Cours et Exercices Corrigés*, Pages Bleus, 2020.

[9] A. P. French, *Vibrations and Waves*, W. W. Norton and Company Inc, New York, 1970.

[10] R. Feynman, R. B. Leighton, M. Sands, M. Bloch, G. Delacôte, *Le Cours de Physique de Feynman: Électromagnétisme, Tome 1*, Addison-Wesley, 2nd edition, 1994.

[11] J. Edminister, *Électromagnétisme : Cours et Problèmes*, Série Schaum, Ediscience International, 2000.

[12] R. Feynman, R. B. Leighton, M. Sands, M. Bloch, G. Delacôte, *Le Cours de Physique de Feynman: Électromagnétisme, Tome 2*, Addison-Wesley, 2nd edition, 1994.

[13] H. J. Pain, *The Physics of Vibrations and Waves*, Wiley, 2005.

[14] H. J. Pain, *Introduction to Vibrations and Waves*, Wiley, 2015.

[15] S. G. Kelly, *Mechanical Vibrations - Theory and Applications*, Cengage Learning, 2011.